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Research Article

Bulanık Çokamaçlı Lineer Kesirli Proğramlama Problemlerinin Çözümleri için Q- Taylor Seri Metodu

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Özet

Bu çalışmada, bulanık çok amaçlı lineer kesirli proğramlama problemlerinin (BÇALPP) çözümleri için q-Taylor seri metodu sunulmuştur. Q-Analizde, q-Taylor serisi q-Türevlerine göre bir fonksiyonun q-Serisine genişlemesidir. Önerilen yaklaşımda, üyelik fonksiyonları parçalı lineer fonksiyonlar olarak tanımlanmaktadır. Q-Taylor serileri kullanılarak dönüştürülen bulanık çok amaçlı lineer kesirli programlama problemleri üyelik fonksiyonları ile birleştirilmiştir. Böylece problem tek bir amaca indirgenmiştir. q-Taylor seri metodunun etkinliğini göstermek için birkaç problemler çözülmüştür.

Anahtar Kelimeler: Bulanık Proğramlama, Bulanık amaçlı hedefler, Çok amaçlı lineer kesirli programlama, Bulanık çok amaçlı lineer kesirli proğramlama, Q-Analiz, Q-Taylor serisi.

Q-Taylor Series Method for Solving Fuzzy Multiobjective Linear Fractional Programming Problem

Abstract

In this work, we present a q-Taylor series method for fuzzy multiobjective linear fractional programming problems (FMOLFPPs). In q-calculus, q-Taylor series is a q-series expansion of a function with respect to q-derivatives. In the proposed approach, membership functions are defined to be piecewise linear. Membership functions associated with each objective of fuzy multiobjective linear fractional programming problem transformed by using q-Taylor series are unified. Thus, the problem is reduced to a single objective. To show the efficiency of the q-Taylor series method, we applied the method to some problems.

Key words: Fuzzy Programming, Fuzzy objective goals, Multiobjective linear fractional programming, Fuzzy multiobjective linear fractional programming, Q-Calculus, Q-Taylor series.

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1. Introduction

The linear fractional programming problem (LFPP), which has been used as an important planning tool in recent years, is applied to different disciplines such as engineering, business, finance, economics, etc. LFPP is generally used for modeling real life problems with one or more objectives such as profit/cost, inventory/sales, actual cost/ standart cost, output/employee etc.

The multiobjective linear fractional programming problem (MOLFPP) is considered in the literature, cf.[11, 12, 16, 18, 21] etc. MOLFPPs pose some computational difficulties, so they are converted into single objective LFPPs and then solved using the metod of Bitran and Novaes [2] or Charnes and Cooper [4]. Uncertainty is an attribute of information [26], and fuzzy set theory has been used for all forms of uncertainty. The model of MOLFPP is reconstructed with fuzzy data. Bellman and Zadeh [1] introduce fuzzy decision-making models in mathematical programming. Luhandjula [17] proposed a linguistic approach to MOLFPP by introducing linguistic variables. A fuzzy approach for solving MOLFPP was presented by Sakawa and Kato [22]. Dutta et al. [8, 9] and Hitosi and Takahashi [14] also studied solutions to FMOLFPPs. A goal programming procedure for fuzzy multiobjective linear fractional programming problem was studied by Pal et al. [19]. On the other hand, many methods of solving fuzzy multiobjective linear programming problems (FMOLFPPs) are available in the literature [3, 7, 13, 24, 25]. They are effective and robust approaches. In [7], polynomial memberships, which are equivalent to fractional membership functions associated with each objective, are obtained by the first order Taylor series.

In this paper, we proposed a new method, the first order q-Taylor series method, for obtaining polynomial membership functions, which are associated with each objective of FMOLFPP.

2. Preliminaries

Definition: If the numerator and denominator in the objective function as well as the constraints are linear, we have a linear fractional programming problem (LFPP) as follows:

$$Optimize \ \frac{cx+\alpha}{dx+\beta}, \tag{2.1}$$

$$s.t.: \ x \in S = \left\{ x \mid Ax \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, \ x \geq 0 \right\}$$

where A is a real $m \times n$ matrix, $b \in R^m$, $x \in R^n$ and S is a nonempty and bounded set. For some values of x, $dx+\beta$ may be equal to zero. To avoid such cases, is generally set to be greater than zero.

Charnes and Cooper [4] showed that if the denominator is constant in sign on the feasible region, the LFPP can be optimized by solving a linear programming problem. Craven [5] , Schaible [23] , Dinkelbach [6] , Gilmore and Gomory [10] and others presented methods for solving LFPPs. However, in many applications, there are two or more conflicting objective functions which are relevant, and some compromise must be sought between them. For example, a management problem may require the profit/cost, quality/cost, and other ratios to be maximized and these conflict. Such types of problems are inherently multiobjective linear fractional programming problems and can be written as:

$$Optimize \ f_k(x) = \frac{c_k x+\alpha_k}{d_k x+\beta_k}, \ k=1,...,K \tag{2.2}$$

$$s.t.: \ x \in S = \left\{ x \mid Ax \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, \ x \geq 0 \right\}$$

where S, A, b and x are as defined in problem (2.1), and $\forall x \in S, \ d_k x+\beta_k > 0$ (k=1,...,K).

In MOLFPP, if an imprecise aspiration level is introduced to each of the objectives then, these fuzzy objectives are termed as fuzzy goals. Let g_k be the aspiration level assigned to the k th objective $Z_k(x)$. Then, the fuzzy goals appear as:

(a) $Z_k(x) \gtrsim g_k$ for maximizing $Z_k(x)$,

(b) $Z_k(x) \lesssim g_k$ for minimizing $Z_k(x)$, where $\gtrsim$ and $\lesssim$ indicate the fuzziness of the aspiration levels, and is to be understood as " essentially more than" and " essentially less than" in the sense of Zimmermann [27].

Hence, the fuzzy multiobjective linear fractional programming problem can be stated as follows:

Find x,

so as to satisfy $Z_k(x) \gtrsim g_k,\ k = 1, 2, ..., k_l$

$$Z_k(x) \lesssim g_k,\ k = k_{l+1}, k_{l+2}, ..., K \tag{2.3}$$

$$s.t.:\ x \in S = \left\{ x \mid Ax \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b,\ x \geq 0 \right\}$$

Now, in the field of fuzzy programming, the fuzzy goals are characterized by their associated membership functions. The membership function μ_k for the kth fuzzy goal $Z_k(x) \gtrsim g_k$ can be expressed algebraically according to Tiwari et al. [24] as:

$$\mu_k(x) = \begin{cases} 0 & \text{if } \ \mathrm{Z}_k(x) \leq l_k \\ \dfrac{\mathrm{Z}_k(x) - l_k}{g_k - l_k} & \text{if } \ l_k \leq \mathrm{Z}_k(x) \leq g_k \\ 1 & \text{if } \ g_k \leq \mathrm{Z}_k(x) \end{cases} \tag{2.4}$$

Where l_k is the lower tolerance limit for the kth fuzzy goal.

On the other hand, the membership function μ_k for the kth fuzzy goal $Z_k(x) \lesssim g_k$ can be defined as:

$$\mu_k(x) = \begin{cases} 0 & \text{if } \ u_k \leq \mathrm{Z}_k(x) \\ \dfrac{u_k - \mathrm{Z}_k(x)}{u_k - g_k} & \text{if } \ g_k \leq \mathrm{Z}_k(x) \leq u_k \\ 1 & \text{if } \ \mathrm{Z}_k(x) \leq g_k \end{cases} \tag{2.5}$$

where u_k is the upper tolerance limit.

Now, in a fuzzy decision environment, the achievement of the objective goals to their aspired levels to the extent possible is actually represented by the possible achievement of their respective membership values to the highest degree.

The relationship between constraints and the objective function(s) in the fuzzy environmentis fully symmetric, that is, there is no longer a difference between the former and the latter [28]. This guarantees the maximization of both objectives' membership values simultaneously. Now we need some basic definitions about q-Taylor series.

Definition: Let $q \in (0,1)$. A q - natural number $[n]_q$ is given by

$$[n]_q := \frac{1 - q^n}{1 - q},\ n \in N \tag{2.6}$$

The factorial of a q -number $[n]_q$ is defined by

$$[0]_q ! := 1,\ [n]_q ! := [n]_q . [n-1]_q ... [1]_q \tag{2.7}$$

q -Pachammer symbol is:

$$(z-a)^{(0)} := 1,\ (z-a)^{(0)} := \prod_{i=0}^{k-1} \left(z - aq^i\right),\ k \in N \tag{2.8}$$

Definition: : Let $f : D \subset R \to R$ be a continuous function. In q -calculus [15], the q -derivative of f is defined by the operatör

$$D_q\left(f(x)\right) := \frac{f(q.x) - f(x)}{(q-1).x},\ x \neq 0,\ q \neq 1, \tag{2.9}$$

$$D_q\left(f(0)\right) := \lim_{x \to 0}\left(D_q\left(f(x)\right)\right). \tag{2.10}$$

Notice that f should be continuous at the point $q.x$ for all $x \in D$ and $q \in (0,1)$.

Definition: Let $f : D \subset R \to R$ be a multivariable continous function, the q-partial derivative of f is given by

$$D_{qx_i} f(x) := \frac{f(Q_i(x)) - f(x)}{(q-1).x_i},\ x_i \neq 0, \tag{2.11}$$

$$x := (x_{1,} x_{2,...,} x_n) \in D,\ i = 1,...,n$$

$$D_{qx_i} f(x)|_{x_i=0} = \lim_{x_i \to 0}(D_{qx_i}(f(x)) \tag{2.12}$$

Where Q_i acting on R^n is an operator defined by

$$Q_i(x_{1,} x_2, ...x_i, ..., x_n) := (x_{1,} x_2, ..., q.x_i, ..., x_n) \tag{2.13}$$

Lemma: Operators $D_{q.x_i}$, $i = 1, 2, ...n$ are R-linear operators.

Definition: Higher order q-partial operator is defined by

$$D^{m+n}_{qx_i^m x_j^n} f(x) := D^m_{qx_i^m}(D^n_{qx_j^n} f(x)) \tag{2.14}$$

where

$$D^{m+n}_{q.x_i^m x_j^n} = D^{m+n}_{q.x_j^n x_i^m}, m, n = 0, 1, 2, ... \tag{2.15}$$

Definition: Let $a = (a_1, a_2, ..., a_n) \in R^n$ be a arbitrary, but fixed and $f : D \subseteq R^n \to R$ be a continous. If f has all the q-partial derivations at a, then the q-differential corresponding to a is defined by

$$d_q f(x,a) = ((x_1 - a_1).D_{q.x_1} + (x_2 - a_2).D_{q.x_2} + ... + (x_n - a_n).D_{q.x_n}) f(a) \tag{2.16}$$

and higher order the q-differential:

$$\begin{aligned} d_q^{(k)} f(x,a) &= ((x_1 - a_1).D_{q.x_1} + (x_2 - a_2)D_{q.x_2} + ... + (x_n - a_n).D_{q.x_n})^{(k)} f(a) \\ &= \sum_{\substack{i_1 + .. + i_n = k \\ i_j \in N}} \left(\frac{[k]_q!}{[i_1]_q!.[i_2]_q!...[i_n]_q!} \right) D^k_{qx_1^{i_1},...,x_n^{i_n}} f(a) \prod_{j=0}^{n} (x_j - a_j)^{(i_j)} \end{aligned} \tag{2.17}$$

Notice that a continous function $f(x)$ in a neighborhood of a that does not include any point with a zero coordinate, has also continous q-partial derivatives.

Lemma: Let $f : D \subset R^n \to R$ be a function having all q-diferentials in some neighborhood of $a \in D$. Then q-Taylor expansion of f at a is given by

$$f(x) = \sum_{k=0}^{\infty} \frac{d_q^k f(x,a)}{[k]_q!} \tag{2.18}$$

Proof: See the proof from [20]

3. Linearization Membership Function Using a Q-Taylor Series Approach

In the FMOLFPP, membership functions associated with each objective are transformed by using q-Taylor series at first, and then a satisfactory values for the variables of the model is obtained by solving the fuzzy model, which has a single objective function. Here, q-Taylor series obtained polynomial membership functions which are equivalent to fractional membership functions associated with each objective. Then, the FMOLFPP can be reduced into a single objective.

The proposed approach can be explained in three steps:

Step 1: Determine $x_k^* = (x_{k1}^*, \ldots, x_{kn}^*)$ which is the value that is used to maximize the k th membership function $\mu_k(x)$ associated with k th objective $Z_k(x)$ $(k = 1, \ldots, K)$ and n is the number of the variables.

Step 2: Transform membership functions by using first-order q-Taylor polynomial series

$$\begin{aligned}\mu_k(x) \cong \hat{\mu}_k(x) &= \sum_{m=0}^{1} \frac{d_q^m \mu_k(x, x_k^*)}{[m]_q!} \\ &= \mu_k(x_k^*) + \left[(x_1 - x_{k1}^*) D_{q_{x_1}} \mu_k(x_k^*) + \ldots + (x_n - x_{kn}^*) D_{q_{x_n}} \mu_k(x_k^*) \right] \\ &= \mu_k(x_k^*) + \sum_{j=1}^{n} (x_j - x_{kj}^*) D_{qx_j} \mu_k(x_k^*)\end{aligned} \tag{3.1}$$

Step 3: Find satisfactory $x^* = (x_1^*, .. x_n^*)$ by solving the reduced problem to a single objective. Note that problem is solved by assuming that weights of the objective are equal. Thus, the problem is written as follows

$$P(x) = \sum_{k=1}^{K} \hat{\mu}_k(x) \tag{3.2}$$

FMOLFP is converted into a new mathematical model. This model is as follows:

$$\begin{aligned}&\max P(x) \\ &\text{s.t } x \in S = \left\{ x \mid Ax \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b,\ x \ge 0 \right\}\end{aligned} \tag{3.3}$$

where

$$\mu_k(x) = \begin{cases} 0 & \text{if} \quad Z_k(x) \le l_k \\ \dfrac{Z_k(x) - l_k}{g_k - l_k} & \text{if} \quad l_k \le Z_k(x) \le g_k \\ 1 & \text{if} \quad g_k \le Z_k(x) \end{cases} \tag{3.4}$$

or

$$\mu_k(x) = \begin{cases} 0 & \text{if} \quad u_k \le Z_k(x) \\ \dfrac{u_k - Z_k(x)}{u_k - g_k} & \text{if} \quad g_k \le Z_k(x) \le u_k \\ 1 & \text{if} \quad Z_k(x) \le g_k \end{cases} \tag{3.5}$$

3.1. Numerical Example

Example: We consider the example studied by Luhandjula and Duran

$$
\begin{aligned}
&Maximize\ Z_1(x) = \frac{x_1 - 4}{-x_2 + 3} \\
&Maximize\ Z_2(x) = \frac{-x_1 + 4}{x_2 + 1} \\
&Subject\ to \qquad -x_1 + 3x_2 \le 0 \\
&\qquad\qquad\qquad\qquad x_1 \le 6 \\
&\qquad\qquad\qquad\quad x_1, x_2 \ge 0
\end{aligned}
$$

fuzzy aspiration levels of the two objectives and the lower tolerance limits of the two fuzzy objective goals are $(g_1, g_2) = (2, 4),\ (l_1, l_2) = (-1, -2)$, respectively,[17, 7]. Find x in order to satisfy the following fuzzy goals:

$$
\begin{aligned}
&Z_1(x) = \frac{x_1 - 4}{-x_2 + 3} \gtrsim 2 \\
&Z_2(x) = \frac{-x_1 + 4}{x_2 + 1} \gtrsim 4 \\
&Subject\ to \qquad -x_1 + 3x_2 \le 0 \\
&\qquad\qquad\qquad\qquad x_1 \le 6 \\
&\qquad\qquad\qquad\quad x_1, x_2 \ge 0
\end{aligned}
\tag{3.6}
$$

The membership functions of the goals are as follows:

$$
\mu_1(x) = \begin{cases} 0 & \text{if} \quad Z_1(x) \le l_1 \\ \dfrac{Z_1(x) - l_1}{g_1 - l_1} & \text{if} \quad l_1 \le Z_1(x) \le g_1 \\ 1 & \text{if} \quad g_1 \le Z_1(x) \end{cases} = \begin{cases} 0 & \text{if} \quad Z_1(x) \le -1 \\ \dfrac{x_1 - x_2 - 1}{-3x_2 + 9} & \text{if} \quad -1 \le Z_1(x) \le 2 \\ 1 & \text{if} \quad 2 \le Z_1(x) \end{cases}
$$

$$
\mu_2(x) = \begin{cases} 0 & \text{if} \quad Z_2(x) \le l_2 \\ \dfrac{Z_2(x) - l_2}{g_2 - l_2} & \text{if} \quad l_2 \le Z_2(x) \le g_2 \\ 1 & \text{if} \quad g_2 \le Z_2(x) \end{cases} = \begin{cases} 0 & \text{if} \quad Z_2(x) \le -2 \\ \dfrac{-x_1 + 2x_2 + 6}{6(x_2 + 1)} & \text{if} \quad -2 \le Z_2(x) \le 4 \\ 1 & \text{if} \quad 4 \le Z_2(x) \end{cases}
$$

If the problem is solved for each of the membership functions one by one then $\mu_1^*(6, 2)$ and $\mu_2^*(0, 0)$. Then membership functions are transformed by using first-order q-Taylor polynomial series for $q = 0.98$:

$$
\mu_1(x) \cong \hat{\mu}_1(x) = \mu_1(6, 2) + \left[(x_1 - 6) D_{qx_1} \mu_1(6, 2) + (x_2 - 2) D_{qx_2} \mu_1(6, 2) \right]
$$

$$
\mu_1(x) \cong \hat{\mu}_1(x) = 0.33x_1 + 0.641x_2 - 2.282
$$

$$
\mu_2(x) \cong \hat{\mu}_2(x) = \mu_2(0, 0) + \left[(x_1 - 0) D_{qx_1} \mu_2(0, 0) + (x_2 - 0) D_{qx_2} \mu_2(0, 0) \right]
$$

where from (2.12)

$$
D_{qx_1} \mu_2(0, 0) = \lim_{(x_1, x_2) \to (0, 0)} (D_{qx_1} \mu_2(x_1, x_2)) = 1
$$

$$D_{qx_2}\mu_2(0,0) = \lim_{(x_1,x_2)\to(0,0)}(D_{qx_2}\mu_2(x_1,x_2)) = 4$$

we get

$$\mu_2(x) \cong x_1 + 4x_2$$

$$P(x) = \hat{\mu}_1(x) + \hat{\mu}_2(x) = 1.33x_1 + 4.641x_2 - 2.282.$$

Thus, the final form of the FMOLFP problem is obtained as follows:

$$\begin{aligned}
&\text{Find } x(x_1, x_2) \text{ so as to}\\
&\textit{Maximize } P(x)\\
&\textit{Subject to}: \; -x_1 + 3x_2 \le 0\\
&\qquad\qquad\qquad x_1 \le 6\\
&\qquad\qquad\qquad x_1, x_2 \ge 0.
\end{aligned}$$

The problem is solved and the solution of the above problem is as follows:

$$x_1 = 6, x_2 = 2 \text{ and } Z_1(x) = 2, Z_2(x) = -2/3$$

Notice that the values of Z_1 and Z_2 above are consistent with (3.6) and membership values are as follows:

$$\mu_1 = 1, \mu_2 = 0.22$$

The membership function values indicate that goals Z_1 and Z_2 are satisfied 100% and 22% respectively, for the obtained solution, which is

$$x_1 = 6, x_2 = 2.$$

4. Conclusions

In this paper we computed the solutions of FMOLFPP using an efficient method which is based on q-calculus theories (in particular, first-order q-Taylor series). Membership functions associated with each objective of the problem are transformed using first-order q-Taylor series.

Actually, FMOLFPP is reduced to MOLPP by first-order q-Taylor series. We assumed that the weights of the objective are equal. Then, the proposed solution method was applied to a numerical example to test the effect of first-order q-Taylor series method with respect to changes in fuzzy aspiration levels, the tolerance limits of problems and q-parameter in q-Taylor series method. The results show that the proposed method is more effective.

References

[1] Bellmann R.E., Zadeh L.A. (1970). Decision making in a fuzzy environment, *Manag. Sci*., (17), 141-164.

[2] Bitran G.R., Novaes A.G. (1973). Linear programming with a fractional objective function, *Operation Research* (21) 22–29.

[3] Chakraborty M., Gupta S. (2002). Fuzzy mathematical programming for multi objective linear fractional programming problem, *Fuzzy Sets and Systems* (125) 335–342.

[4] Charnes A., Cooper W. (1962). Programming with linear fractional functions, *Naval Research Logistics Quarterly* (9) 181-186.

[5] Craven B.D. (1988). *Fractional Programming*, Heldermann Verlag, Berlin,

[6] Dinkelbach W. (1967). On nonlinear fractional programming, *Manage. Sci.* (13) 492–498.

[7] Toksari, M. D. (2008). Taylor series approach to fuzzy multiobjective linear fractional programming, *Information Sciences* (178) 1189–1204

[8] Dutta D., Tiwari R.N., Rao J.R. (1993). Fuzzy approaches for multiple criteria linear fractional optimization: a comment, *Fuzzy Sets and Systems* (54) 347–349.

[9] Dutta D., Tiwari R.N., Rao J.R. (1992). Multiple objective linear fractional programming a fuzzy set theoretic approach, *Fuzzy Sets and Systems* (52) 39–45.

[10] Gilmore P.C., Gomory R.E., (1963). A linear programming approach to the cutting stock problem. *Oper. Res*. (11) 863–888.

[11] Gupta P., Bhatia D. (2001). Sensitivity analysis in fuzzy multiobjective linear fractional programming problem, *Fuzzy Sets and Systems* (122) 229–236.

[12] Guzel N., Sivri M. (2005). Taylor series solution of multiobjective linear fractional programming problem, *Trakya University Journal Science* (6) 80–87.

[13] Hannan E.L., (1981). Linear programming with multiple fuzzy goals, *Fuzzy Sets and Systems* (6) 235–248.

[14] Hitosi M.S., Takahashi Y.J. (1992). Pareto optimality for multiobjective linear fractional programming problems with fuzzy parameters, *Information Sciences* (63) 33–53.

[15] Kac V., Cheung P. (2002). *Quantum Calculus*, Springer, New York,

[16] Kornbluth J.S.H., Steuer R.E. (1981). Multiple objective linear fractional programming, *Management Science* (27) 1024–1039.

[17] Luhandjula M.K. (1984). Fuzzy approaches for multiple objective linear fractional optimization, *Fuzzy Sets and Systems* (13) 11–23.

[18] Nykowski I., Zolkiski Z. (1985). A compromise procedure for the multiple objective linear fractional programming problem, *European Journal of Operational Research* (19) 91–97.

[19] Pal B.B., Moitra B.N., Maulik U. (2003). A goal programming procedure for fuzzy multiobjective linear fractional programming problem, *Fuzzy Sets and Systems* (139) 395–405.

[20] Rajkovic P.M., Stankovic M.S., Marinkovic S.D. (2003). On q-iterative methods for solving equations and systems. *Novi Sad J.Math* (33) 127-137.

[21] Saad O. (2007). On stability of proper efficient solutions in multiobjective fractional programming problems under fuzziness, *Mathematical and Computer Modelling* (45) 221–231.

[22] Sakawa M., Kato K. (1988). Interactive decision-making for multiobjective linear fractional programming problems with block angular structure involving fuzzy numbers, *Fuzzy Sets and Systems* (97) 19–31.

[23] Schaible S. (1976). Fractional programming I: duality, *Manage. Sci.* (22) 658–667.

[24] Tiwari R.N., Dharmar S., Rao J.R. (1987). Fuzzy goal programming an additive model, *Fuzzy Sets and Systems* (24) 27–34.

[25] Cevikel A. C., Ahlatcioglu M. (2010). Solutions for fuzzy matrix games, *Computer& Mathematics with Applications*, (60) 399-410.

[26] Zadeh L. (2005). Toward a generalized theory of uncertainty (GTU) an outline, *Information Sciences* (172) 1–40.

[27] Zimmermann H.J. (1978). Fuzzy programming and linear programming with several objective functions, *Fuzzy Sets and Systems* (1) 45–55.

[28] Zimmermann H.J. (1987). *Fuzzy Set Theory and its Applications*, Kluwer Academic Publishers, Boston.