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Research Article

A Note on the Stability Analysis of Nonlinear Fractional Difference Equations: Comparative Approach

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Abstract

In this study, we focus on nonlinear forward fractional difference equations with order $\nu \in (0,1]$ and construct stability analysis regarding h-stability and Mittag-Leffler stability notions. The main results of the paper are obtained by equiparating the equation in the spotlight with an auxiliary fractional difference equation. The outcomes of the manuscript provide an alternative approach to the ongoing theory of discrete fractional equations since the method used in the main results deviates from the fundamental tools of stability theory, namely fixed point theory and Liapunov's direct method.

Keywords: h-stability, Mittag-Leffler stability, Forward fractional difference equation, Perturbed equation.

Doğrusal Olmayan Kesirli Fark Denklemlerinin Kararlılık Analizi Üzerine Bir Not: Karşılaştırmalı Yaklaşım

Öz

Bu çalışmada $\nu \in (0,1]$ mertebesinde doğrusal olmayan kesirli fark denklemleri üzerinde durulmuş olup, h-kararlılık ve Mittag-Leffler kararlılığı kavramları kullanılarak bir kararlılık analizi yapılmıştır. Makalenin temel sonuçları odaklanılan kesirli fark denkleminin yardımcı bir kesirli fark denklemi ile karşılaştırılması ve kıyaslanması ile elde edilmiştir. Bu çalışmanın çıktıları literatürde kesirli denklemlerin kararlılık analizinde genellikle kullanılan sabit nokta teorisi ve Liapunov teorisi gibi araçların dışında bir yol kullanılarak elde edildiği için halen gelişmekte olan ayrık kesirli denklemlerin teorisine farklı bir bakış açısı sunarak katkı sağlamıştır.

Anahtar Kelimeler: h-Kararlılık, Mittag-Leffler kararlılık, Kesirli fark denklemi, Pertürbe denklem.

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1. Introduction

Noninteger-order equations, namely fractional equations, have become a voguish research area in the last two decades due to their potential application in theoretical and applied sciences. The popularity of the subject has directed researchers from a wide range of disciplines to construct fractional analogs of real-life models, and this eventuates excellent applications of fractional equations. By a quick literature review, one may easily find out the utilization of fractional equations in neural networks, signal processing, mechanics, biology, medicine, finance, and economics.

The qualitative theory of fractional equations is a fruitful and developing research area for mathematicians, and undoubtfully these types of equations are appreciably studied on continuous and discrete-time domains. Consequentially, the theories for fractional differential and difference equations have been enhanced correspondingly. As it is well known, one of the landmark branches of qualitative theory of differential and difference equations is the stability theory, and naturally, stability analysis of fractional equations on continuous and discrete-time domains is extensively studied. We refer to readers (Baleanu, Wu, Bai, & Chen, 2017; Choi, Kang, & Koo, 2014; Choi & Koo, 2011; Kang & Koo, 2019) as the remarkable studies for stability analysis of fractional differential equations, and also we indicate the inspiring papers (Chen, 2011; Chen & Liu, 2012; Wyrwas & Mozyrska, 2015) for the stability analysis of discrete fractional equations. In the present work, we aim to contribute to the already established literature regarding stability analysis of fractional equations by focusing on nonlinear forward fractional difference equations and providing sufficient conditions for the stability of the solutions. Indeed, the main objective of the paper is two-fold:

(I) ***h*-stability:** As the first objective, we examine the h-stability of the solutions for nonlinear fractional difference equations and their perturbations. The notion of h-stability was initially introduced as an extension of the exponential stability definition for solutions of differential equations by Pinto (1984). By h-stability of the zero solution for the initial value problem

$$\begin{cases} x'(t) = f(t,x) \\ x(t_0) = x_0 \end{cases}, \qquad f(t,0) = 0,$$

we mean the solution x satisfies $|x(t)| \leq c|x_0| \frac{h(t)}{h(t_0)}$ where $c \geq 1$ and h is a bounded function. Discussion of h-stability is also carried to discrete-time domains. We refer to (Choi, Koo, & Song, 2004; Medina, 1998; Medina & Pinto, 1996) as related studies. Besides, this definition is adopted to fractional equations defined on continuous-time by Choi et al. (2014).

(II) **Mittag-Leffler stability:** Secondly, we investigate the sufficient conditions for the Mittag-Leffler stability of the nonlinear forward fractional difference equations and their perturbations. It should be noted that the Mittag-Leffler function is primarily proposed in the paper (Mittag-Leffler, 1902) and has become essential for fractional equations. It is possible to establish an analogy between the exponential functions in differential and difference equations and Mittag-Leffler functions in fractional calculus since Mittag-Leffler functions are used as a fractional exponential function. On the other hand, Mittag-Leffler stability is proposed via the Mittag-Leffler function, and the Mittag-Leffler stability for the solutions of fractional equations is vastly studied on both continuous and discrete-time domains.

In our analysis, we are inspired by the papers (Choi, Koo, & Ryu, 2003; Choi et al., 2014; Choi & Koo, 2011; Choi et al., 2004) and invert an inequality which is crucial to conduct a comparative approach between two nonlinear fractional discrete initial value problems. To the best of our knowledge, such a correlative approach has not been performed in discrete fractional calculus. Thus, the main results of this paper are distinguished from the stability results in the existing literature regarding fractional difference equations since the outcomes of the manuscript avoid utilization of the fixed point theory or Liapunov's direct method.

The organization of the manuscript is as follows: In the next section, we provide a summary for forward discrete fractional calculus. Section 3 is devoted to the presentation of our stability results regarding h-stability and Mittag-Leffler stability.

2. Essentials for forward discrete fractional calculus

This chapter is devoted to the presentation of basic definitions and results on forward fractional discrete calculus. Given definitions and results can be found in (Atici & Eloe, 2007; Atıcı & Eloe, 2009b; Atici & Eloe, 2015).

Let Γ stand for the conventional gamma function. First, we introduce the following notation

$$t^{(\mu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}.$$

Note that if $t+1-\mu \in \{0,-1,\cdots,-k,\cdots\}$, then we set $t^{(\mu)} = 0$. For the readership, it is convenient to list the following identities: Assume that the factorial functions are well defined. Then

- **i.** $(t-\mu)t^{(\mu)} = t^{(\mu+1)}$ for $\mu \in \mathbb{R}$
- **ii.** $\mu^{(\mu)} = \Gamma(\mu+1)$,
- **iii.** If $t \leq r$, then $t^{(\nu)} \leq r^{(\nu)}$ for any $\nu > r$,
- **iv.** If $0 < \nu < 1$, then $t^{(\alpha\nu)} \geq (t^{(\alpha)})^{\nu}$,
- **v.** $t^{(\alpha+\beta)} = (t-\beta)^{(\alpha)} t^{(\beta)}$.

In the sequel, the forward fractional summation is given by

$$\Delta_a^{-\nu} f(t) = \sum_{s=a}^{t-\nu} \frac{\big(t-\sigma(s)\big)^{(\nu-1)}}{\Gamma(\nu)} f(s),$$

where $\sigma(s) = s+1$, $\nu \geq 0$, and $a \in \mathbb{R}$. The notation $\mathbb{N}_a$ indicates the set $\{a, a+1, a+2, \cdots\}$, and obviously, the operator $\Delta_a^{-\nu}$ maps $\mathbb{N}_a$ to $\mathbb{N}_{a+\nu}$. Also, we write $\Delta^{-\nu} f(t)$ when $a = 0$. Next, we present the Riemann-Liouville forward fractional difference as follows:

$$\Delta_a^{\mu} f(t) = \Delta_a^{m-\nu} f(t) = \Delta^{m}\big(\Delta_a^{-\nu} f(t)\big),$$

where $\mu > 0$, $m-1 < \mu \leq m$ for positive integer m, and $-\nu = \mu - m$. We shall recall the following properties regarding the Riemann-Liouville forward fractional difference:

- **vi.** $\Delta t^{(\mu)} = \mu t^{(\mu-1)}$,
- **vii.** If $\mu \neq -1$ and $\mu+\nu+1$ is not a non-positive integer, then

$$\Delta_\mu^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{(\mu+\nu)}.$$

Furthermore, we list some auxiliary results on forward fractional discrete calculus.

Theorem 2.1: Let f be a real valued function defined on $\mathbb{N}_a$ and suppose that $\mu, \nu > 0$. Then

$$\Delta^{-\nu}\big(\Delta^{-\mu} f(t)\big) = \Delta^{-\nu-\mu} f(t) = \Delta^{-\mu}\big(\Delta^{-\nu} f(t)\big).$$

Theorem 2.2: For any function f defined on $\mathbb{N}_a$ and $\nu > 0$, the equation,

$$\Delta^{-\nu}\big(\Delta f(t)\big) = \Delta(\Delta^{-\nu}) f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a).$$

As it is highlighted in Remark 2.1 of (Atıcı & Eloe, 2009b), if one replaces ν with $\nu+1$ and utilizes Theorem 1, the following identity,

$$\Delta^{-\nu} f(t) = \Delta^{-\nu-1}\big(\Delta f(t)\big) + \frac{(t-a)^{(\nu)}}{\Gamma(\nu+1)} f(a)$$

is straightforward.

Theorem 2.3: For any $\nu \in \mathbb{R}$ and positive integer p, the following equality holds

$$\Delta^{-\nu}\Delta^{p} f(t) = \Delta^{p}\Delta^{-\nu} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{(\nu-p+k)}}{\Gamma(\nu-p+k+1)} \Delta^{k} f(a) \qquad (1)$$

where f is defined on $\mathbb{N}_a$.

Theorem 2.4: Let p be a positive integer and $\nu > p$. Then

$$\Delta^{p}(\Delta^{-\nu} f(t)) = \Delta^{-(\nu-p)} f(t).$$

The following result, which appears as Lemma 2.1 in (Atıcı & Eloe, 2009a), establishes a linkage between forward and backward fractional summation operators.

Lemma 2.5: Let $0 \le m-1 < \nu \le m$ where m is an integer, a be a positive integer and $y(t)$ be defined on $\mathbb{N}_a$. Then the following identities hold:

(1)$\Delta_a^{\nu} y(t-\nu) = \nabla_a^{\nu} y(t)$ for $t \in \mathbb{N}_{m+a}$.

(2)$\Delta_a^{-\nu} y(t+\nu) = \nabla_a^{-\nu} y(t)$ for $t \in \mathbb{N}_a$.

3. Main Results

In this part, we concentrate on the forward fractional discrete initial value problem

$$\begin{cases} \Delta^{\nu} x(t) = f(t+\nu-1, x(t+\nu-1)) \\ x(0) = x_0 \end{cases}, \qquad (2)$$

and its perturbation

$$\begin{cases} \Delta^{\nu} z(t) = f(t+\nu-1, z(t+\nu-1)) + g(t+\nu-1, z(t+\nu-1)) \\ z(0) = z_0 \end{cases}, \qquad (3)$$

where $t \in \mathbb{N}_{1-\nu}$, $\nu \in (0,1]$, and $f(t,0) = g(t,0) = 0$. With reference to Lemma 2.4 of (Chen, Luo, & Zhou, 2011) and Lemma 2.2 of (Chen, 2011), we express the solutions $x: \mathbb{N} \to \mathbb{R}$ and $z: \mathbb{N} \to \mathbb{R}$ of nonlinear fractional equations (2) and (3) as

$$x(t) = x_0 + \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} \big(t-\sigma(s)\big)^{(\nu-1)} f\big(s+\nu-1, x(s+\nu-1)\big), \qquad (4)$$

and

$$z(t) = z_0 + \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} \big(t-\sigma(s)\big)^{(\nu-1)} \big[f\big(s+\nu-1, z(s+\nu-1)\big) + g\big(s+\nu-1, z(s+\nu-1)\big)\big], \qquad (5)$$

respectively. We pursue two approaches for stability, namely h-stability and Mittag-Leffler stability, for the zero solutions of problems (2) and (3). To achieve this task, we employ an inequality that enables us to conduct a fractional comparison between (2) & (3) and auxiliary fractional difference equations and obtain sufficient conditions to ensure stability.

Inspired by (Choi et al., 2003), we prove the following lemma, which is crucial for the setup of our stability results.

Lemma 3.1: Let the function $f(t,r)$ be nonnegative and nondecreasing in its second argument for any fixed $t \in \mathbb{N}$. Suppose that the nonnegative functions x and y satisfy

$$x(t) - \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} \big(t-\sigma(s)\big)^{(\nu-1)} f\big(s, x(s)\big) < y(t) - \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} \big(t-\sigma(s)\big)^{(\nu-1)} f\big(s, y(s)\big). \qquad (6)$$

If $x(0) < y(0)$, then $x(t) < y(t)$ for all $t \in \mathbb{N}$.

Proof: Suppose that f is nonnegative, nondecreasing in its second argument, and the inequality (6) holds. Also, we assume that $x(0) < y(0)$. To obtain a contradiction, we suppose $x(t^*) = y(t^*)$, and $x(s) < y(s)$ for $0 \le s < t^*$. By using (6), we write

$$x(t^*) < y(t^*) + \frac{1}{\Gamma(\nu)} \left[\sum_{s=1-\nu}^{t^*-\nu} \big(t-\sigma(s)\big)^{(\nu-1)} f\big(s, x(s)\big) - \sum_{s=1-\nu}^{t^*-\nu} \big(t-\sigma(s)\big)^{(\nu-1)} f\big(s, y(s)\big) \right]$$

$$\le y(t^*),$$

where we used the monotonicity of the function f. This is a contradiction, and the proof is complete. □

3.1. *h*-Stability

We shall introduce the notion of h-stability due to Definition 12 of (Choi et al., 2014) on fractional calculus as the initial step.

Definition 3.2: The zero solution of the forward fractional difference equation given in (2) is said to be h-stable if there exist a constant $c \ge 1$ and a bounded, positive function $h: \mathbb{N} \to \mathbb{R}$ such that

$$|x(t)| \le c|x(a)| \frac{h(t)}{h(a)}, t \ge a \ge 0,$$

for $|x(a)| < \delta$.

Next, we present our first result regarding the h-stability of the zero solution for the initial value problem (2).

Theorem 3.3: Suppose that the function f in (2) satisfies

$$|f(t,x)| \leq q(t,|x|), \tag{7}$$

where $q: \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing in its second argument and $q\,(t,0) = 0$. Moreover, consider the following forward fractional difference equation

$$\begin{cases} \Delta^\nu y(t) = q(t+\nu-1, y(t+\nu-1)) \\ \qquad\qquad y(0) = y_0 \end{cases},\ t \in \mathbb{N}_{1-\nu} \tag{8}$$

as the auxiliary problem. If the zero solution of (8) is h-stable, then the zero solution of (2) is also h-stable whenever $y_0 > |x_0|$.

Proof: Let x be the solution of (2), and suppose that condition (7) holds. By (4), we write

$$\begin{aligned}|x(t)| &\leq |x_0| + \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} \left(t-\sigma(s)\right)^{(\nu-1)} \left|f\left(s+\nu-1, x(s+\nu-1)\right)\right| \\ &\leq |x_0| + \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} \left(t-\sigma(s)\right)^{(\nu-1)} q(s+\nu-1, |x(s+\nu-1)|).\end{aligned} \tag{9}$$

Assuming that $|x_0| < y_0$, we have

$$|x(t)| - \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} \left(t-\sigma(s)\right)^{(\nu-1)} q(s+\nu-1, |x(s+\nu-1)|)$$

$$\leq |x_0| < y_0 = y(t) - \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} \left(t-\sigma(s)\right)^{(\nu-1)} q\left(s+\nu-1, y(s+\nu-1)\right),$$

which leads to $|x(t)| < y(t)$ for all $t \in \mathbb{N}$ by Lemma 3.1. If the solution y of the auxiliary problem (8) is h-stable, i.e.,

$$y(t) \leq c^* |y(a)| \frac{h(t)}{h(a)}, t \geq a \geq 0,$$

for $|y(a)| < \delta$, then

$$|x(t)| < y(t) \leq c^* |y(a)| \frac{h(t)}{h(a)} = c|x(a)| \frac{h(t)}{h(a)},$$

where $y(a) = k|x(a)|$ and $c = kc^*$ with $k > 1$. This completes the proof. □

Subsequently, we aim to present another result regarding h-stability for the perturbed equation (3). For this purpose, first, we have to derive a nonlinear variation of parameters formula, which is instrumental in obtaining the solution $z(t,x_0)$ of (3) in terms of the solution $x(t,x_0)$ of the problem (2). The nonlinear variation of parameters formula for perturbed nabla fractional difference equations and the dependence of solutions on initial conditions are studied in detail by the papers (Mohan, 2013) and (Deekshitulu & Mohan, 2013). It should be highlighted that the nabla fractional difference equation

$$\begin{cases} \nabla^\nu x(n+1) = f(n, x(n)) \\ \qquad x(0) = x_0 \end{cases}, n \in \mathbb{N}, \nu \in (0,1]$$

examined in (Deekshitulu & Mohan, 2013) and the forward fractional difference equation (2) are identical due to Lemma 2.5. More explicitly, we have $\nabla^\nu x(n+1) = \Delta^\nu x(n+1-\nu)$, and a substitution $t = n+\nu-1$ yields the desired identity. Inspired by the papers mentioned above, we aspire to adapt the nonlinear variation of parameters formula given in (Mohan, 2013; Deekshitulu & Mohan, 2013) for (3) to propose sufficient conditions for the h-stability of its zero solution.

We provide the following crucial result by rewriting Theorem 2.2 of (Deekshitulu & Mohan, 2013) in terms of forward fractional calculus to obtain the variation of parameters for the nonlinear perturbed equation in (3). The proof is omitted since it is tantamount to the proof of Theorem 2.2 by Deekshitulu and Mohan (2013).

Theorem 3.4: Suppose that the partial derivative $\frac{\partial f}{\partial x}$ exists for the function $f: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. Let $x(t,x_0)$ be the solution of the initial value problem (2) and set

$$H(t+\nu-1) = \frac{\partial}{\partial x} f\left(t+\nu-1, x(t+\nu-1)\right).$$

Then

$$\Phi(t+\nu-1, x_0) = \frac{\partial}{\partial x_0} x(t+\nu-1, x_0) \tag{10}$$

exists, and it is the solution to the initial value problem

$$\begin{cases} \Delta^\nu \Phi(t,x_0) = H(t+\nu-1)\Phi(t+\nu-1,x_0) \\ \qquad\qquad \Phi(0,x_0) = I \end{cases},\ t \in \mathbb{N}. \tag{11}$$

The next result enables us to construct a comparative stability result for the solution of (3) since it describes how to express the solution of the perturbed problem (3) in accordance with the unperturbed equation in (2). It should be emphasized that the following outcome is originated from Theorem 3.1 of (Deekshitulu & Mohan, 2013), and its proof is presented by following the similar steps of the principal result established for nabla fractional difference equations.

Theorem 3.5 (Nonlinear variation of parameters formula): Let $f(t,x)$ and $g(t,x)$ be two functions, and assume that $\frac{\partial f}{\partial x}$ exists and continuous. Also, consider the function Φ which is defined as in (10). Then any solution $z(t) = z(t,x_0)$ of the problem (3) with the updated initial condition $z(0) = x_0$ satisfies the equation

$$\begin{aligned} z(t,x_0) = x\Bigg(t, x_0 + \frac{1}{\Gamma(\nu)} \sum_{k=0}^{t-\nu} \Big[\xi^{-1}\left(k+\nu, w(k), w(k+\nu)\right) \\ \sum_{s=1-\nu}^{k} \left(k+\nu-\sigma(s)\right)^{(\nu-1)} g\left(s+\nu-1, z(s+\nu-1)\right)\Big]\Bigg) \end{aligned} \tag{12}$$

where

$$\xi\left(k+\nu, w(k), w(k+\nu)\right) = \int_0^1 \Phi\left(k+\nu, rw(k+\nu) + (1-r)w(k)\right) dr.$$

Proof: Consider the forward fractional difference equation in (3) with the initial condition $z(0) = x_0$. Then, the solution z of the perturbed problem is given by

$$\begin{aligned} z(t+\nu, x_0) = x_0 + \frac{1}{\Gamma(\nu)} \sum_{s=1-\nu}^{t-\nu} \Big(\left(t+\nu-\sigma(s)\right)^{(\nu-1)} \\ [f(s+\nu-1, z(s+\nu-1,x_0)) + g(s+\nu-1, z(s+\nu-1,x_0))]\Big) \end{aligned} \tag{13}$$

due to (5). To propose the variation of parameters formula for the forward fractional perturbed equation, we shall determine a function so that $z(t,x_0) = x(t,w(t))$ with $w(0) = x_0$. Once the desired identity $z(t,x_0) = x(t,w(t))$ is achieved, one may easily write

$$x(t+\nu,w(t+\nu))=x_0+\frac{1}{\Gamma(\nu)}\sum_{s=1-\nu}^{t}\left(\left(t+\nu-\sigma(s)\right)^{(\nu-1)}\right.$$
$$\left.[f(s+\nu-1,x(s+\nu-1,w(s)))+g(s+\nu-1,x(s+\nu-1,w(s)))]\right) \tag{14}$$

with the help of (13). Since $x(t,x_0)$ is the solution to problem (2), it is clear that

$$x\big(t+\nu,w(t)\big)=x_0+\frac{1}{\Gamma(\nu)}\sum_{s=1-\nu}^{t}\left(t+\nu-\sigma(s)\right)^{(\nu-1)}f\left(s+\nu-1,x\big(s+\nu-1,w(s)\big)\right). \tag{15}$$

By using (14) and (15), we have

$$\frac{1}{\Gamma(\nu)}\sum_{s=1-\nu}^{t}\left(t+\nu-\sigma(s)\right)^{(\nu-1)}g\left(s+\nu-1,x\big(s+\nu-1,w(s)\big)\right)$$
$$=x\big(t+\nu,w(t+\nu)\big)-x\big(t+\nu,w(t)\big). \tag{16}$$

We employ the mean value theorem on the right-hand side of (16) and obtain

$$\int_0^1\frac{\partial}{\partial x_0}x\big(t+\nu,rw(t+\nu)+(1-r)w(t)\big)dr$$
$$=\frac{x\big(t+\nu,w(t+\nu)\big)-x\big(t+\nu,w(t)\big)}{w(t+\nu)-w(t)},$$

where $\frac{\partial}{\partial x_0}$ indicates the partial derivative of the function with respect to its second argument. This yields

$$\big(w(t+\nu)-w(t)\big)\int_0^1\frac{\partial}{\partial x_0}x\big(t+\nu,rw(t+\nu)+(1-r)w(t)\big)dr$$
$$=\frac{1}{\Gamma(\nu)}\sum_{s=1-\nu}^{t}\left(t+\nu-\sigma(s)\right)^{(\nu-1)}g\big(s+\nu-1,z(s+\nu-1,x_0)\big).$$

For the sake of brevity, we set

$$\xi(t+\nu,w(t),w(t+\nu))=\int_0^1\Phi(t+\nu,rw(t+\nu)+(1-r)w(t))\,dr,$$

then write

$$w(t+\nu)-w(t)=\frac{\xi^{-1}\big(t+\nu,w(t),w(t+\nu)\big)}{\Gamma(\nu)}\sum_{s=1-\nu}^{t}\left[\left(t+\nu-\sigma(s)\right)^{(\nu-1)}\right.$$
$$\left.g\big(s+\nu-1,z(s+\nu-1,x_0)\big).\right] \tag{17}$$

Taking the sum of both sides provides the following equation

$$w(t)=x_0+\frac{1}{\Gamma(\nu)}\sum_{k=0}^{t-\nu}\left[\xi^{-1}\big(k+\nu,w(k),w(k+\nu)\big)\right.$$
$$\left.\sum_{s=1-\nu}^{k}\left(k+\nu-\sigma(s)\right)^{(\nu-1)}g\big(s+\nu-1,z(s+\nu-1)\big)\right]. \tag{18}$$

Consequentially, the desired equation $z(t,x_0)=x\big(t,w(t)\big)$ holds whenever w is as in (18). The proof is complete. □

Remark 3.6: To provide an alternative representation of (12), we consider

$$\frac{x\big(t,w(t)\big)-x(t,x_0)}{w(t)-x_0}=\int_0^1\frac{\partial}{\partial x_0}x(t,rw(t)+(1-r)x_0)dr$$

due to the mean value theorem. This implies

$$x\big(t,w(t)\big)=x(t,x_0)+(w(t)-x_0)\xi(t,w(t),x_0)$$

which results in

$$z(t,x_0)=x(t,x_0)+\frac{1}{\Gamma(\nu)}\sum_{k=0}^{t-\nu}\left[\xi(t,w(t),x_0)\xi^{-1}\big(k+\nu,w(k),w(k+\nu)\big)\right.$$
$$\left.\sum_{s=1-\nu}^{k}\left(k+\nu-\sigma(s)\right)^{(\nu-1)}g\big(s+\nu-1,z(s+\nu-1)\big)\right].$$

Now, we are all set to present our next stability result based on the h-stability of the zero solution of perturbed forward fractional difference equation with the initial condition $z(0)=x_0$.

Theorem 3.7: Suppose that the zero solution of the problem (2) is h-stable with nonincreasing function h and the solution Φ of (11) is bounded, i.e., $n\le|\Phi(t,x_0)|\le N$ for all $t\in\mathbb{N}$. Additionally, we assume

$$g(t,z)\le q(t,|z|), \tag{19}$$

Where $q:\mathbb{N}\times\mathbb{R}_+\to\mathbb{R}_+$ is strictly increasing in its second argument, and $q(t,0)=0$. Consider the following forward fractional difference equation

$$\begin{cases}\Delta^\nu u(t)=c^*q(t+\nu-1,u(t+\nu-1)),\ t\in\mathbb{N}_{1-\nu}\\ u(0)=u_0\end{cases}. \tag{20}$$

If the zero solution of (20) is h-stable, then the zero solution of the perturbed equation in (3) with $z(0)=x_0$ is also h-stable when $u_0=c^*|x_0|$ for $c^*\ge1$.

Proof: Suppose that the zero solution of (2) is h-stable and (19) is satisfied. First of all, we write

$$z(t,x_0)=x(t,x_0)+\frac{1}{\Gamma(\nu)}\sum_{k=0}^{t-\nu}\left[\xi(t,w(t),x_0)\xi^{-1}\big(k+\nu,w(k),w(k+\nu)\big)\right.$$
$$\left.\sum_{s=1-\nu}^{k}\left(k+\nu-\sigma(s)\right)^{(\nu-1)}g\big(s+\nu-1,z(s+\nu-1)\big)\right]$$

by Remark 1.

Then, we consider

$$|z(t,x_0)|\le|x(t,x_0)|$$
$$+\frac{1}{\Gamma(\nu)}\sum_{k=0}^{t-\nu}\left[\left|\xi(t,w(t),x_0)\xi^{-1}\big(k+\nu,w(k),w(k+\nu)\big)\right|\right.$$
$$\left.\sum_{s=1-\nu}^{k}\left(k+\nu-\sigma(s)\right)^{(\nu-1)}\left|g\big(s+\nu-1,z(s+\nu-1)\big)\right|\right]$$

$$\leq c|x_0| + \frac{1}{\Gamma(\nu)}\frac{N}{n}\sum_{s=1-\nu}^{t-\nu}\big(t-\sigma(s)\big)^{(\nu-1)} q(s+\nu-1, |z(s+\nu-1)|)$$

$$\leq c^*|x_0| + c^*\frac{1}{\Gamma(\nu)}\sum_{s=1-\nu}^{t-\nu}\big(t-\sigma(s)\big)^{(\nu-1)} q(s+\nu-1, |z(s+\nu-1)|), \tag{21}$$

where $c^* = \max\{c, \frac{N}{n}\}$. As an implementation of (21), we get

$$|z| - c^*\frac{1}{\Gamma(\nu)}\sum_{s=1-\nu}^{t-\nu}\big(t-\sigma(s)\big)^{(\nu-1)} q(s+\nu-1, |z(s+\nu-1)|)$$

$$\leq c^*|x_0| = u_0$$

$$= u(t) - c^*\frac{1}{\Gamma(\nu)}\sum_{s=1-\nu}^{t-\nu}\big(t-\sigma(s)\big)^{(\nu-1)} q\big(s+\nu-1, u(s+\nu-1)\big).$$

Then by Lemma 3.1, $|z(t)| < u(t)$ for all $t \in \mathbb{N}$. To conclude, we write the following due to the condition regarding h-stability of the zero solution for (20)

$$|z(t)| < u(t) \leq \hat{c}u_0\frac{h(t)}{h(0)} = d|x_0|\frac{h(t)}{h(0)},$$

where $d = \hat{c}c^* \geq 1$. Therefore, the zero solution of the perturbed equation is h-stable. □

3.2. Mittag-Leffler Stability

In this part, we examine the Mittag-Leffler stability of the solutions of the forward fractional difference equations given in (2) and (3). First, we present the discrete counterpart of the Mittag-Leffler function given in (Abdeljawad, 2011) and the stability notion for fractional difference equations in the sense of Mittag-Leffler due to (Wyrwas & Mozyrska, 2015; Choi & Koo, 2011).

Definition 3.8: For $\lambda \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{C}$ (Re $\lambda > 0$), the discrete Mittag-Leffler function is defined by

$$E_{(\alpha,\beta)}(\lambda,\gamma) = \sum_{k=0}^{\infty}\lambda^k\frac{\big(\gamma+(k-1)(\alpha-1)\big)^{(k\alpha)}\big(\gamma+k(\alpha-1)\big)^{(\beta-1)}}{\Gamma(\alpha k+\beta)}. \tag{22}$$

For $\beta = 1$, (22) turns to

$$E_{(\alpha)}(\lambda,\gamma) = E_{(\alpha,1)}(\lambda,\gamma) = \sum_{k=0}^{\infty}\lambda^k\frac{\big(\gamma+(k-1)(\alpha-1)\big)^{(k\alpha)}}{\Gamma(\alpha k+1)}.$$

Remark 3.9: It should be highlighted that there is another Mittag-Leffler function used in the existing literature, and the following representation does not contradict Definition 2 given in (Abdeljawad, 2011). The alternative form of the Mittag-Leffler function is given by

$$E_{(\alpha,\beta)}(\lambda,\gamma) = \sum_{k=0}^{\infty}\lambda^k\binom{\gamma-k+\alpha k+\beta-1}{\gamma-k},$$

and if $\beta = 1$, then

$$E_{(\alpha)}(\lambda,\gamma) = E_{(\alpha,1)}(\lambda,\gamma) = \sum_{k=0}^{\infty}\lambda^k\binom{\gamma-k+\alpha k}{\gamma-k}.$$

Next, we give the following definition based on the discrete counterpart of the Mittag-Leffler function, which can be found in (Wyrwas & Mozyrska, 2015; Choi & Koo, 2011).

Definition 3.10: The solution of the initial value problem (2) is said to be Mittag-Leffler stable if

$$|x(t)| \leq \Big(m(x_0)E_{(\alpha)}(\lambda,t)\Big)^b,$$

where $\lambda < 0, \alpha \in (0,1], m(0) = 0, m(x) \geq 0$, and m is locally Lipschitz with Lipschitz constant m_0.

Theorem 3.11: Consider the forward fractional difference equation (2), and let us recall the auxiliary equation given in (8)

$$\begin{cases}\Delta^\nu y(t) = q(t+\nu-1, y(t+\nu-1)) \\ \qquad y(0) = y_0\end{cases},$$

where $q: \mathbb{N}\times\mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing in its second argument and $q\,(t,0) = 0$. Additionally, suppose that condition (7), i.e.,

$$|f(t,x)| \leq q(t,|x|)$$

holds. If the zero solution of the problem (8) is Mittag-Leffler stable, then the zero solution of (2) is also Mittag-Leffler stable when $y_0 > |x_0|$.

Proof: Consider (2) and suppose that condition (7) holds. If we use (4) together with condition (7), then the inequality (9) is straightforward, as it is done in the proof of Theorem 3.3. By assuming $y_0 > |x_0|$, one may easily deduce that $|x(t)| < y(t)$ for all $t \in \mathbb{N}$ due to Lemma 3.1. Sequentially, we have

$$\begin{aligned}|x(t)| < y(t) &\leq \Big(m(y_0)E_{(\alpha)}(\lambda,t)\Big)^b \\ &\leq \Big(m_0y_0E_{(\alpha)}(\lambda,t)\Big)^b \\ &= \Big(m_0d|x_0|E_{(\alpha)}(\lambda,t)\Big)^b \\ &= \Big(m^*(|x_0|)E_{(\alpha)}(\lambda,t)\Big)^b, t \in \mathbb{N},\end{aligned}$$

where $\lambda < 0, y_0 = d|x_0|$ with $d > 1$, $m^*(|x_0|) = m_0d|x_0|$, and m^* is locally Lipschitz. Thus the zero solution of (2) is Mittag-Leffler stable. □

Theorem 3.12: Suppose that the zero solution of the problem (2) is h-stable with nonincreasing function h and the solution Φ of (11) is bounded, i.e., $n < |\Phi(t,x_0)| \leq N$ for all $t \in \mathbb{N}$. Additionally, we assume that the condition (19), that is,

$$g(t,z) \leq q(t,|z|),$$

holds where the function q is strictly increasing in its second argument and $q\,(t,0) = 0$. Also, we consider the forward fractional difference equation given in (20), i.e.,

$$\begin{cases}\Delta^\nu u(t) = c^*q(t+\nu-1, u(t+\nu-1)), \ t \in \mathbb{N}_{1-\nu} \\ \qquad u(0) = u_0\end{cases}$$

If the zero solution of (20) is Mittag-Leffler stable, then the zero solution of the perturbed equation in (3) with $z(0) = x_0$ is also Mittag-Leffler stable when $u_0 > |x_0|$.

Proof: The proof can be completed on the grounds of the proofs of Theorem 3.7 and Theorem 3.11. Therefore, we omit the proof. □

4. Conclusion

In the stability theory of fractional equations, fixed point theory and Liapunov's direct method can be regarded as the classical instruments to analyze the qualitative behavior of solutions. In our work, we aim to pursue a comparative approach as an alternative to these conventional tools of stability analysis and study h-stability and Mittag-Leffler stability of forward fractional difference equations. This paper handles nonlinear forward fractional difference equations in the form of (2) and their perturbation in (3). The setup of the paper relies on the introduction of an auxiliary fractional difference equation and inquiring about its stability. This enables us to obtain stability results for the equations (2) and (3) without restrictive conditions adopted in the standard methods. Thus, we illustrate an alternative path in the qualitative analysis of discrete fractional equations and contribute to the established literature.

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