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Research Article

On Lacunary Convergent Complex Uncertain Triple Sequences Determined by Orlicz Function

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Abstract

In this study, we acquire some new class of lacunary convergent triple sequences by utilizing the notions of uncertainty theory and Orlicz function. We examine some topological features of the established sequence spaces and obtain significant results.

Keywords: Almost convergent sequence, Orlicz function, Lacunary sequence, Complex uncertain variable.

Orlicz Fonksiyonu Tarafından Belirlenen Karmaşık Belirsiz Üç İndisli Dizilerin Lacunary Yakınsaklığı Üzerine

Öz

Bu çalışmada, belirsizlik teorisi ve Orlicz fonksiyonu kavramları kullanılarak bazı yeni lacunary yakınsak üç indisli dizilerin sınıfını elde ettik. Kurulan dizi uzaylarının bazı topolojik özelliklerini inceledik ve önemli sonuçlar elde ettik.

Anahtar Kelimeler: Hemen hemen yakınsak dizi, Orlicz fonksiyonu, Lacunary dizi, Kompleks karmaşık değişken.

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1. Introduction

Theory of uncertainty was firstly originated by Liu [7]. Liu defined convergence notions of sequences of uncertain variables in the same study. The notions of uncertainty theory have been examined from different aspects with applications. Complex uncertain sequence spaces of uncertain variables were worked by Tripathy and Nath [10]. After then, many researchers studied on this topic (see [1,2,6,11]). The concept of lacunary sequences was firstly given by Freedman et al. [3]. The aim of this study is to investigate lacunary convergent complex uncertain triple sequences with regards to (w.r.t.) an Orlicz function in uncertainty space.

Orlicz function is a continuous, non-decreasing and convex function defined as $M:[0,\infty)\to[0,\infty)$ with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\to\infty$ as $x\to\infty$.

Lindenstrauss and Tzafriri [4] utilized the Orlicz function to create the sequence space:

$$l_M=\left\{y=\{y_p\}\in w:\sum_{p=1}^{\infty}M\left(\frac{|y_p|}{\rho}\right)<\infty,\qquad \text{for some } \rho>0\right\},$$

where *w* indicates the class of all sequences.

The space l_M with the norm

$$\|y\|=\inf\left\{\rho>0:\sum_{p=1}^{\infty}M\left(\frac{|y_p|}{\rho}\right)\leq 1\right\},$$

becomes a Banach space which is named an Orlicz sequence space. Recently, several authors investigated several Orlicz sequences space (see [8 − 9].

We now present a brief of the uncertainty theory. Let L be a σ −algebra on a non-empty set Γ.

A set function M is named an uncertain measure if it supplies the subsequent axioms:

A1: $M\{\Gamma\}=1$;

A2: $M\{\Lambda\}+M\{\Lambda^c\}=1$, for any $\Lambda\in L$,

A3: For each countable sequence of $\{\mu_j\}\in L$,

we obtain

$$M\left(\bigcup_{j=1}^{\infty}\mu_j\right)\leq\sum_{j=1}^{\infty}M\{\mu_j\}.$$

The triplet (Γ,L,M) is named an uncertainty space, and every element $\Lambda\in L$ is said an event.

2. Material and Method

With the description in the introduction, it can be observed that this study is qualitative with grounded theory method. Papers [6] and [7] supply concepts of convergence of uncertains sequences and also [2], [5] and [9] provide a fundamental survey of the lacunary convergence concepts of uncertain sequences.

By utilizing the notions of uncertainty theory and Orlicz function, we acquire new class of lacunary convergent triple sequences.

3. Results and Discussion

Kişi and Ünal [5] and Tripathy and Nath [10] examined the notions of statistical convergence and lacunary statistical convergence for complex uncertain sequence.

Now, we give the lacunary sequence notions of uncertain triple sequences w.r.t. Orlicz function and examine the relations among them.

Definition 3.1. *The complex uncertain triple sequence (γ_{jkl}) is named to be lacunary strongly convergent almost surely (a.s.) to σ w.r.t. an Orlicz function M provided that there is an event Λ with $M\{\Lambda\}=1$ such that*

$$\lim_{r,s,t\to\infty}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}}M\|\gamma_{jkl}(\tau)-\sigma(\tau)\|=0,$$

for all $\tau\in\Lambda$.

Definition 3.2. *The sequence (γ_{jkl}) is named to be lacunary strongly convergent in measure to σ w.r.t. an Orlicz function M provided that*

$$\lim_{r,s,t\to\infty}M\left[\left\{\tau\in\Gamma:\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}}M\|\gamma_{jkl}(\tau)-\sigma(\tau)\|>\rho\right\}\right]=0,$$

for each $\rho>0$.

Definition 3.3. *The sequence (γ_{jkl}) is named to be lacunary strongly convergent in mean to σ w.r.t. an Orlicz function M provided that*

$$\lim_{r,s,t\to\infty}E\left[\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}}M\|\gamma_{jkl}(\tau)-\sigma(\tau)\|\right]=0.$$

Definition 3.4. *Assume $\Phi,\Phi_{jkl},\ldots$ be the complex uncertainty distribution of complex uncertain variables γ,γ_{jkl} respectively. Then, the sequence (γ_{jkl}) lacunary strong convergent in distribution to σ w.r.t. an Orlicz function M provided that*

$$\lim_{r,s,t\to\infty}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}}M\|\Phi_{jkl}(z)-\Phi(z)\|=0$$

for all complex z at which $\Phi(z)$ is continuous.

We acquire the relations among the given notions below.

Theorem 3.1. *When the sequence (γ_{jkl}) lacunary strongly convergent in mean to σ w.r.t. an Orlicz function M, then (γ_{jkl}) lacunary strongly convergent in measure to σ.*

Proof. With the aid of Markov inequality that for any $\rho > 0$, we obtain

$$\lim_{r,s,t\to\infty} M\left[\left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left\|\gamma_{jkl}(\tau) - \sigma(\tau)\right\| > \rho\right\}\right] \le \lim_{r,s,t\to\infty} \frac{E\left[\frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left\|\gamma_{jkl}(\tau) - \sigma(\tau)\right\|\right]}{\rho} = 0.$$

Thus, (γ_{jkl}) lacunary strongly converges in measure to σ w.r.t. an Orlicz function M.

The converse of the theorem is not true. This can be indicated with the subsequent example.

Example 3.1. Take into consideration the uncertainty space (Γ, L, M). It becomes $\Gamma = \{\tau_1, \tau_2, \tau_3, ...\}$ with

If

$$\sup_{\tau_{r+s+t}\in\Lambda} \frac{1}{r+s+t+1} < 0.5$$

then

$$\mathrm{M}(\Lambda) = \sup_{\tau_{r+s+t}\in\Lambda} \frac{1}{r+s+t+1};$$

if

$$\sup_{\tau_{r+s+t}\in\Lambda^c} \frac{1}{r+s+t+1} < 0.5$$

Then

$$M(\Lambda) = 1 - sup_{\tau_{r+s+t}\in\Lambda^c} \frac{1}{r+s+t+1};$$

otherwise $M(\Lambda) = 0.5$.

Consider uncertain variables as

$$\gamma_{rst}(\tau) = \begin{cases} (r+s+t+1)i, & if\ \tau = \tau_{r+s+t}, \\ 0, & \text{otherwise.} \end{cases}$$

for $r, s, t \in \mathbb{N}$ and $\gamma(\tau) \equiv 0$. For $\rho > 0$, we acquire

$$\lim_{r,s,t\to\infty} M\left[\left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left\|\gamma_{jkl}(\tau) - \sigma(\tau)\right\| > \rho\right\}\right]$$
$$= \lim_{r,s,t\to\infty} M\left[\left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left\|\gamma_{jkl}(\tau)\right\| > \rho\right\}\right]$$
$$= \lim_{r,s,t\to\infty} M(\{\tau_{jkl}\}) = \lim_{r,s,t\to\infty} \frac{1}{j+k+l+1} = 0$$

as $(j,k,l) \in I_{rst}$. The sequence $\{\gamma_{rst}\}$ lacunary strongly converges in measure to σ.

But for each $r, s, t \ge 2$, we get the uncertainty distribution of uncertain variable $\|\gamma_{rst} - \sigma\| = \|\gamma_{rst}\|$ is

$\Phi_{rst}(x) = 0$, where $x < 0$,

$\Phi_{rst}(x) = 1 - \frac{1}{r+s+t+1}$; where $0 \le x < r+s+t+1$,

$\Phi_{rst}(x) = 1$, otherwise.

$$E\left[\frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left\|\gamma_{\mathrm{jkl}}(\tau) - \sigma(\tau)\right\|\right] = \int_0^{r+s+t+1} 1 - \left(1 - \frac{1}{r+s+t+1}\right) dx = 1.$$

Namely, the $\{\gamma_{rst}(\tau)\}$ does not converge in mean to $\sigma(\tau)$ w.r.t. an Orlicz function.

Theorem 3.2. *Assume the complex uncertain triple sequence* $\{\gamma_{rst}\}$ *where* $\{\alpha_{rst}\}$ *is the real part and* $\{\beta_{rst}\}$ *is the imaginary part, for* $r, s, t \in \mathbb{N}$. *When the sequences* $\{\alpha_{rst}\}$ *and* $\{\beta_{rst}\}$ *lacunary strongly convergent in measure to* σ_1 *and* σ_2 *respectively w.r.t an Orlicz function M, then the sequence* $\{\gamma_{rst}\}$ *lacunary strongly uniformly convergent in distribution to* $\sigma = \sigma_1 + i\sigma_2$.

Proof. The complex uncertainty distribution Φ must have a certain point of continuity $w = q + it$. For any $\vartheta > q, \kappa > t$, we get

$$\begin{aligned} \{\alpha_{rst} \le q, \beta_{rst} \le t\} &= \{\alpha_{rst} \le q, \beta_{rst} \le t, \sigma_1 \le \vartheta, \sigma_2 \le \kappa\} \\ &\cup \{\alpha_{rst} \le q, \beta_{rst} \le t, \sigma_1 \le \vartheta, \sigma_2 > \kappa\} \\ &\cup \{\alpha_{rst} \le q, \beta_{rst} \le t, \sigma_1 > \vartheta, \sigma_2 > \kappa\} \\ &\cup \{\alpha_{rst} \le q, \beta_{rst} \le t, \sigma_1 > \vartheta, \sigma_2 \le \kappa\} \\ &\subseteq \{\sigma_1 \le \vartheta, \sigma_2 \le \kappa\} \\ &\cup \{M\|\alpha_{rst}(\tau) - \sigma_1(\tau)\| \ge \vartheta - q\} \\ &\cup \{M\|\beta_{rst}(\tau) - \sigma_2(\tau)\| \ge \kappa - t\} \end{aligned}$$

With the aid of subadditivity axiom

$$\Phi_{jkl}(w) = \Phi_{jkl}(q+it) \leq \Phi(q+it) + M\{\tau \in \Gamma: M\|\alpha_{rst}(\tau) - \sigma_1(\tau)\| \geq \vartheta - q\} + M\{\tau \in \Gamma: \|\beta_{rst}(\tau) - \sigma_2(\tau)\| \geq \kappa - t\}.$$

As $\{\alpha_{rst}\}$ and $\{\beta_{rst}\}$ lacunary strongly convergent in measure to σ_1 and σ_2 respectively w.r.t Orlicz function *M*, so for $\rho > 0$ and $(j,k,l) \in I_{rst}$

$$\lim_{r,s,t\to\infty} \mathrm{M}\left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\|\alpha_{rst}(\tau) - \sigma_1(\tau)\| \geq \vartheta - q > \rho\right\} = 0$$

and

$$\lim_{r,s,t\to\infty} M\left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\|\beta_{rst}(\tau) - \sigma_2(\tau)\| \geq \kappa - t > \rho\right\} = 0.$$

As a result, we get

$$\lim_{r,s,t\to\infty} \sup \Phi_{jkl}(w) \leq \Phi(q+it).$$

for any $\vartheta > q, \kappa > t$. Taking $\vartheta + i\kappa \to q + it$, we obtain $\lim_{r,s,t\to\infty} \sup \Phi_{jkl}(w) \leq \Phi(w)$.

On the other hand, for any $x < q, y < t$ we acquire

$$\begin{aligned}\{\sigma_1 \leq x, \sigma_2 \leq y\} &= \{\alpha_{rst} \leq q, \beta_{rst} \leq t, \sigma_1 \leq x, \sigma_2 \leq y\} \\ &\cup \{\alpha_{rst} > q, \beta_{rst} \leq t, \sigma_1 \leq x, \sigma_2 \leq y\} \\ &\cup \{\alpha_{rst} \leq q, \beta_{rst} > t, \sigma_1 \leq x, \sigma_2 \leq y\} \\ &\cup \{\alpha_{rst} > q, \beta_{rst} > t, \sigma_1 \leq x, \sigma_2 \leq y\} \\ &\subseteq \{\alpha_{rst} \leq q, \beta_{rst} \leq t\} \\ &\cup \{M\|\alpha_{rst} - \sigma_1\| > q - x\} \\ &\cup \{M\|\beta_{rst} - \sigma_2\| > t - y\}\end{aligned}$$

which means

$$\Phi_{jkl}(x+iy) \leq \Phi(q+it) + M\{\tau \in \Gamma: M\|\alpha_{rst}(\tau) - \sigma_1(\tau)\| \geq q - x\} + M\{\tau \in \Gamma: \|\beta_{rst}(\tau) - \sigma_2(\tau)\| \geq t - y\}.$$

Since

$$\lim_{r,s,t\to\infty} M\left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\|\alpha_{rst}(\tau) - \sigma_1(\tau)\| \geq q - x > \rho\right\} = 0$$

and

$$\lim_{r,s,t\to\infty} M\left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\|\beta_{rst}(\tau) - \sigma_2(\tau)\| \geq t - y > \rho\right\} = 0,$$

we get

$$\Phi(x+iy) \leq \lim_{r,s,t\to\infty} \inf \Phi_{jkl}(q+it)$$

for any $x < q, y > t$. Taking $x + iy \to q + it$, we obtain

$$\Phi(w) \leq \lim_{r,s,t\to\infty} \inf \Phi_{jkl}(w).$$

From the relations

$$\lim_{r,s,t\to\infty} \sup \Phi_{jkl}(w) \leq \Phi(w)$$

and $\Phi(w) \leq \lim_{r,s,t\to\infty} \inf \Phi_{jkl}(w)$ that $\Phi_{jkl}(w) \to \Phi(w)$ as $r,s,t \to \infty$ and $(j,k,l) \in I_{rst}$. Namely, $\{\gamma_{rst}\}$ lacunary strongly uniformly convergent in distribution to $\sigma = \sigma_1 + i\sigma_2$.

Theorem 3.3. *If (γ_{jkl}) is lacunary strongly convergent uniformly a.s. to σ w.r.t. an Orlicz function M, then (γ_{jkl}) is lacunary strongly convergent in measure to σ w.r.t. that Orlicz function M.*

Proof. If (γ_{jkl}) lacunary strongly convergent uniformly a.s. to σ w.r.t. an Orlicz function M, then

$$\lim_{r,s,t\to\infty} M\left(\bigcup_{(j,k,l)\in I_{rst}} \left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\|\gamma_{jkl}(\tau) - \sigma(\tau)\| > \rho\right\}\right) = 0.$$

But

$$M\left(\left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left\|\gamma_{jkl}(\tau)-\sigma(\tau)\right\| > \rho\right\}\right)$$
$$\leq M\left(\bigcup_{(j,k,l)\in I_{rst}} \left\{\tau \in \Gamma: \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left\|\gamma_{jkl}(\tau)-\sigma(\tau)\right\| > \rho\right\}\right).$$

So, (γ_{jkl}) is lacunary strongly convergent in measure to σ w.r.t. that Orlicz function M.

Here, we present some new classes of sequences of uncertain variables utilizing an Orlicz functions. Now, we present the fundamental features of the spaces $[N_{\theta_{r,s,t}}, M]_0, [N_{\theta_{r,s,t}}, M]_1, [N_{\theta_{r,s,t}}, M]_\infty$ and acquire some significant results.

Theorem 3.4. *The complex uncertain sequence classes*

$$[N_{\theta_{r,s,t}}, M]_0, [N_{\theta_{r,s,t}}, M]_1, [N_{\theta_{r,s,t}}, M]_\infty$$

are linear spaces.

Proof. We demonstrate the consequence for the class of complex uncertain sequence $[N_{\theta_{r,s,t}}, M]_\infty$.

The other situations can be showed in the similar manner. Assume $(\gamma_{jkl}), (\mu_{jkl}) \in [N_{\theta_{r,s,t}}, M]_\infty$ and $\alpha, \beta \in \mathbb{C}$. Then, there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_{rst} \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)\|}{\rho_1}\right) < \infty$$

and

$$\sup_{rst} \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\mu_{jkl}(\tau)\|}{\rho_2}\right) < \infty.$$

Take $\rho_3 = \max(2\alpha\rho_1, 2\beta\rho_2)$.

As M is non-decreasing and convex, we get

$$\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\alpha\gamma_{jkl}(\tau)+\beta\mu_{jkl}(\tau)\|}{\rho_3}\right)$$
$$\leq \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\alpha\gamma_{jkl}(\tau)\|}{\rho_3}\right) + \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\alpha\mu_{jkl}(\tau)\|}{\rho_3}\right)$$
$$\leq \sum_{(j,k,l)\in I_{rst}} \frac{1}{2}\left[M\left(\frac{\|\gamma_{jkl}(\tau)\|}{\rho_1}\right) + M\left(\frac{\|\mu_{jkl}(\tau)\|}{\rho_2}\right)\right]$$
$$\leq \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)\|}{\rho_1}\right) + \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\mu_{jkl}(\tau)\|}{\rho_2}\right)$$
$$\Rightarrow \sup_{rst} \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\alpha\gamma_{jkl}(\tau)+\beta\mu_{jkl}(\tau)\|}{\rho_3}\right)$$
$$\leq \sup_{rst} \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)\|}{\rho_1}\right)$$
$$+ \sup_{rst} \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\mu_{jkl}(\tau)\|}{\rho_2}\right) < \infty.$$

So, we obtain $\alpha\gamma_{jkl}(\tau) + \beta\mu_{jkl}(\tau) \in [N_{\theta_{r,s,t}}, M]_\infty$. As a result $[N_{\theta_{r,s,t}}, M]_\infty$ is a linear space.

Theorem 3.5. *The spaces* $[N_{\theta_{r,s,t}}, M]_0, [N_{\theta_{r,s,t}}, M]_1$ *and* $[N_{\theta_{r,s,t}}, M]_\infty$ *are solid.*

Proof. Take $(\gamma_{jkl}) \in [N_{\theta_{r,s,t}}, M]_0$. Then, there is a $\rho > 0$ such that

$$\lim_{r,s,t\to\infty} \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)\|}{\rho}\right) = 0.$$

Assume (α_{jkl}) be a sequence of scalars such that $|\alpha_{jkl}| \leq 1$. Then, for all r, s, t we get

$$\frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\alpha_{jkl}\gamma_{jkl}(\tau)}{\rho}\right) \leq \frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\gamma_{jkl}(\tau)}{\rho}\right).$$

Therefore, we obtain

$$\frac{1}{h_{rst}} \sum_{(j,k,l)\in I_{rst}} M\left(\frac{\alpha_{jkl}\gamma_{jkl}(\tau)}{\rho}\right) = 0,$$

so, $(\alpha_{jkl}\gamma_{jkl}(\tau)) \in [N_{\theta_{r,s,t}}, M]_0$.

Theorem 3.6. *For any Orlicz function* $M, [N_{\theta_{r,s,t}}, M]_\infty$ *is a normed linear space whose norm is determined by*

$$g(\gamma) = inf\left\{\rho > 0: sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}\|}{\rho}\right) \leq 1, \quad r,s,t = 1,2,3,\ldots\right\}.$$

The infimum is taken over all $\rho > 0$.

Proof. Obviously, $g(\gamma_{jkl}) \geq 0$, for all $(\gamma_{jkl}) \in [N_{\theta_{r,s,t}}, M]_\infty$ and $g(\bar{\theta}) = 0$. Assume $\rho_1, \rho_2 > 0$ be such that

$$\sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)\|}{\rho_1}\right) \leq 1,$$

$$\sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\mu_{jkl}(\tau)\|}{\rho_2}\right) \leq 1.$$

Take $\rho = \rho_1 + \rho_2$. Then, we obtain

$$\begin{aligned}
&\sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)+\mu_{jkl}(\tau)\|}{\rho}\right)\\
&\leq \sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)+\mu_{jkl}(\tau)\|}{\rho_1+\rho_2}\right)\\
&\leq \sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} \frac{\rho_1}{\rho_1+\rho_2} M\left(\frac{\|\gamma_{jkl}(\tau)\|}{\rho_1}\right)\\
&+\frac{\rho_2}{\rho_1+\rho_2} M\left(\frac{\|\mu_{jkl}(\tau)\|}{\rho_2}\right)\\
&\leq \frac{\rho_1}{\rho_1+\rho_2}\sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)\|}{\rho_1}\right)\\
&+\frac{\rho_2}{\rho_1+\rho_2}\sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\mu_{jkl}(\tau)\|}{\rho_2}\right) \leq 1.
\end{aligned}$$

As the ρ's are not negative, so we get,

$$\begin{aligned}
&g\left(\gamma_{jkl}(\tau)+\mu_{jkl}(\tau)\right)\\
&= \inf\left\{\rho > 0: \sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)+\mu_{jkl}(\tau)\|}{\rho}\right) \leq 1\right\}\\
&= \inf\left\{\rho_1 > 0: \sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)\|}{\rho_1}\right) \leq 1\right\}\\
&+ \inf\left\{\rho_2 > 0: \sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\mu_{jkl}(\tau)\|}{\rho_2}\right) \leq 1\right\}\\
&= g\left(\gamma_{jkl}(\tau)\right) + g\left(\mu_{jkl}(\tau)\right)
\end{aligned}$$

As a result

$$g\left(\gamma_{jkl}(\tau)+\mu_{jkl}(\tau)\right) = g\left(\gamma_{jkl}(\tau)\right) + g\left(\mu_{jkl}(\tau)\right).$$

Now, $\xi \in \mathbb{C}$, suppose $\xi \neq 0$, then

$$\begin{aligned}
g\left(\xi\gamma_{jkl}(\tau)\right) &= \inf\left\{\rho > 0: \sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\xi\gamma_{jkl}(\tau)\|}{\rho}\right) \leq 1\right\}\\
&= \inf\left\{|\xi|r > 0: \sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\xi\gamma_{jkl}(\tau)\|}{r}\right) \leq 1\right\}; r = \frac{\rho}{|\xi|}\\
&= \inf\left\{r > 0: \sup_{rst}\frac{1}{h_{rst}}\sum_{(j,k,l)\in I_{rst}} M\left(\frac{\|\gamma_{jkl}(\tau)\|}{r}\right) \leq 1\right\}\\
&= |\xi| g\left(\gamma_{jkl}(\tau)\right).
\end{aligned}$$

This finalizes the proof

Theorem 3.7. *The spaces $[N_{\theta_{r,s,t}}, M]_0$, $[N_{\theta_{r,s,t}}, M]_1$ and $[N_{\theta_{r,s,t}}, M]_\infty$ are monotone.*

4. Conclusions and Recommendations

This study establishes the various lacunary strongly convergent notions of the triple sequences of uncertain variable with regards to Orlicz function. The consequence obtained here generalizes the existing conclusions.

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