Araştırma Makalesi

Research Article

Investigating Spherical Images of a Curve According to Type-1 Bishop Frame in Weyl Space Using Prolonged Covariant Derivative

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Abstract

In this study, we investigated spherical images of a curve according to type-1 Bishop frame in three dimensional Weyl space. Further, we expressed the relations among Frenet-Serret and type-1 Bishop frame apparatus. We defined the concepts of general helix, slant helix, spherical curve and also circle by using prolonged covariant derivative in Weyl space. Later, provided that these spherical images satisfy the above definitions, the conditions obtained were expressed in terms of first and second curvatures and hence Bishop curvatures. Additionally, parallel displacement condition of the binormal vector fields of the n_1 and n_2 Bishop spherical images of a curve along their own tangent vector fields was discussed.

Keywords: Weyl space, General helix, Slant helix, Spherical curve, Spherical image.

Prolonged Kovaryant Türevi Kullanarak Weyl uzayındaki Bir Eğrinin Tip-1 Bishop Çatısına Göre Küresel Resimlerinin İncelenmesi

Öz

Bu çalışmada, Weyl uzayındaki bir eğrinin tip-1 Bishop çatısına göre küresel resimlerini inceledik. Ayrıca, Frenet-Serret ve tip-1 Bishop çatı aparatları arasındaki bağıntıları ifade ettik. Prolonged kovaryant türevi kullanarak, Weyl uzayında genel helis, slant helis, küresel eğri ve ayrıca çember kavramlarını tanımladık. Daha sonra, bu küresel resimlerin yukarıdaki tanımları sağlaması halinde, elde edilen şartlar birinci ve ikinci eğrilikler ve dolayısıyla Bishop eğrilikleri cinsinden ifade edildi. Bunlara ek olarak, bir eğrinin n_1 ve n_2 Bishop küresel resimlerinin binormal vektör alanlarının kendi teğet vektör alanları boyunca paralel kayma şartı ele alındı.

Anahtar Kelimeler: Weyl uzayı, Genel helis, Slant helis, Küresel eğri, Küresel resim.

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1. Introduction

Bishop frame (or type-1 Bishop frame) was introduced by Bishop (1975). This frame was also named as alternative or parallel frame of the curves. Many researchers used Bishop frame in several spaces, such as Bükçü and Karacan (2008a and 2009), Yılmaz et al. (2010; in Euclidean space), Bükçü and Karacan (2008b; in Lorentzian space), Karacan and Bükçü (2007 and 2008), Yılmaz (2009; in Minkowski 3-space) and Kofoğlu (2020; in Weyl space).

2. Preliminaries

Let $C: x^i = x^i(s)$ (s is the arc length parameter of C) be a curve in three dimensional Weyl space W_3 ($i = 1,2,3$). Let us denote Frenet-Serret frame and Bishop (or type-1 Bishop) frame belonging to C by $\{v_1, v_2, v_3\}$ and $\{v_1, n_1, n_2\}$, respectively. Both of these frames are orthonormal bases.

Frenet-Serret formulas are expressed as

$$v_1^k \dot{\nabla}_k v_1^i = \kappa_1 v_2^i$$
$$v_1^k \dot{\nabla}_k v_2^i = -\kappa_1 v_1^i + \kappa_2 v_3^i \quad (1)$$
$$v_1^k \dot{\nabla}_k v_3^i = -\kappa_2 v_2^i$$

where κ_1 and κ_2 are the first and second curvatures of C, respectively.

Derivative formulas of the vector fields of Bishop frame are in the following form:

$$v_1^k \dot{\nabla}_k v_1^i = k_1 n_1^i + k_2 n_2^i$$
$$v_1^k \dot{\nabla}_k n_1^i = -k_1 v_1^i \quad (2)$$
$$v_1^k \dot{\nabla}_k n_2^i = -k_2 v_1^i$$

Here, k_1 and k_2 are Bishop curvatures (Bishop, 1975). Their equivalents in Weyl space (Kofoğlu, 2020) are

$$k_1 = \overset{p}{T_k} v_1^k v_p^i n_1^j g_{ij} \quad (j,k,p = 1,2,3) \quad (3)$$

or

$$k_1 = \overset{p}{\underset{1p}{2}} v_1^i n_1^j g_{ij} = c_1^i n_1^j g_{ij} \quad (4)$$

and

$$k_2 = \overset{p}{T_k} v_1^k v_p^i n_2^j g_{ij} \quad (5)$$

or

$$k_2 = \overset{p}{\underset{1p}{2}} v_1^i n_2^j g_{ij} = c_1^i n_2^j g_{ij} \quad (6)$$

where, $\overset{p}{\underset{1}{2}} = \overset{p}{T_k} v_1^k$ is geodesic curvature of the net (v_1, v_2, v_3) (Tsareva, 1990) and $c_1^i = \overset{p}{\underset{1p}{2}} v^i$ is the geodesic curvature vector field of the net (v_1, v_2, v_3) (Tsareva, 1990).

Also, $k_1 = \kappa_1 \cos\theta$, $k_2 = \kappa_1 \sin\theta$, $\kappa_1^2 = k_1^2 + k_2^2$ and $\kappa_2 = v_1^k \dot{\nabla}_k \theta$ ($\theta = \theta(s)$) where $\theta = \sphericalangle(v_2^i, n_1^i)$ (Kofoğlu, 2020).

There is the following relation among the vector fields of these two frames (Kofoğlu, 2020):

$$\begin{pmatrix} v_1^i \\ v_2^i \\ v_3^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_1^i \\ n_1^i \\ n_2^i \end{pmatrix} \quad (7)$$

3. The Expression of Special Curves in Weyl Space

Definition 1. Let C be a be a curve in three dimensional Weyl space. C is called a general helix if the tangent vector field v_1 of C has constant angle φ with some fixed vector field u, i.e.,

$$g_{ij} v_1^i u^j = \cos\varphi = constant$$

where $g_{ij} u^i u^j = 1$.

Using Şemin (1983), we can express the condition to be a general helix in the following form:

Theorem 1. C is a general helix if and only if

$$\frac{\kappa_2}{\kappa_1} = constant$$

where κ_1 and κ_2 are the first and second curvatures of C.

With the help of Izumiya and Takeuchi (2004) and Kula et al. (2010) and using prolonged covariant derivative, the following proposition can be given:

Proposition 1. If C is a slant helix,

$$\frac{\kappa_1^2}{(\kappa_1^2 + \kappa_2^2)^{3/2}} \left(v_1^k \dot{\nabla}_k \frac{\kappa_2}{\kappa_1} \right) = constant$$

is satisfied.

Proof. Let C be a slant helix. Then C is a curve with $\kappa_1 \neq 0$ and $\left(v_1^k \dot{\nabla}_k \frac{\kappa_2}{\kappa_1} \right)$ is a constant function.

Let $\bar{C} : y^i = y^i(\bar{s})$ be the spherical image of principal normal vector field v_2 of C ($\bar{s}$ is the arc length parameter of $\bar{C}$). Then, using Frenet frame the following equalities are satisfied:

$$v_1^k \dot{\nabla}_k y^i = v_1^k \dot{\nabla}_k v_2^i$$
$$\left(\bar{v}_1^k \dot{\bar{\nabla}}_k y^i \right) a = -\kappa_1 v_1^i + \kappa_2 v_3^i$$
$$\bar{v}_1^i = -\frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}} v_1^i + \frac{\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}} v_3^i \quad (8)$$

where $\bar{v}_1$ is the tangent vector field of $\bar{C}$ and $a = a(s) = \dfrac{1}{\sqrt{\kappa_1^2 + \kappa_2^2}}$.

The prolonged covariant derivative of $\underset{1}{\bar{v}^i}$ **in the direction of** $\underset{1}{v}$ is

$$
\begin{aligned}
\underset{1}{v^k}\dot{\nabla}_k\underset{1}{\bar{v}^i} &= \left(\underset{1}{\bar{v}^k}\dot{\bar{\nabla}}_k\underset{1}{\bar{v}^i}\right)a \\
\underset{1}{\bar{v}^k}\dot{\bar{\nabla}}_k\underset{1}{\bar{v}^i} &= \bar{\kappa}_1\underset{2}{\bar{v}^i} = \underset{1}{\bar{c}^i} \\
&= \frac{\kappa_2\kappa_1^2}{(\kappa_1^2+\kappa_2^2)^2}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{\kappa_2}{\kappa_1}\right)\underset{1}{v^i} \\
&+ \frac{\kappa_1^3}{(\kappa_1^2+\kappa_2^2)^2}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{\kappa_2}{\kappa_1}\right)\underset{3}{v^i} - \underset{2}{v^i}
\end{aligned}
\tag{9}
$$

where $\bar{\kappa}_1$ is the first curvature of $\bar{C}$, $\underset{2}{\bar{v}}$ is the principal normal vector field of $\bar{C}$ and $\underset{1}{\bar{c}^i}$ is the geodesic curvature vector field of the net $(\underset{1}{\bar{v}}, \underset{2}{\bar{v}}, \underset{3}{\bar{v}})$.

The geodesic curvature of the spherical image of the principal normal vector field of $\underset{2}{v}$ is

$$
\kappa_g^2 = g_{ij}\underset{1}{\bar{c}^i}\underset{1}{\bar{c}^j} = \bar{\kappa}_1^2 = \left[\underset{1}{v^k}\dot{\nabla}_k\frac{\kappa_2}{\kappa_1}\right]^2\frac{\kappa_1^4}{(\kappa_1^2+\kappa_2^2)^3} + 1.
\tag{10}
$$

Let us denote the first term of in the right hand side of the above equality by σ^2. Then, we get

$$
\bar{\kappa}_1^2 = \sigma^2 + 1.
\tag{11}
$$

Since the spherical image of the principal normal vector field $\underset{2}{v}$ is a part of a circle in S^2, $\bar{\kappa}_1$ must be non-zero constant and so

$$
\sigma^2 = \left[\underset{1}{v^k}\dot{\nabla}_k\frac{\kappa_2}{\kappa_1}\right]^2\frac{\kappa_1^4}{(\kappa_1^2+\kappa_2^2)^3}
$$

or

$$
\sigma = \left(\underset{1}{v^k}\dot{\nabla}_k\frac{\kappa_2}{\kappa_1}\right)\frac{\kappa_1^2}{(\kappa_1^2+\kappa_2^2)^{3/2}}
\tag{12}
$$

is a constant function.

Here $\kappa_1 = g_{ij}\left(\underset{1}{v^k}\dot{\nabla}_k\underset{1}{v^i}\right)\underset{2}{v^j} = \underset{1}{\overset{2}{c}}$ and $\kappa_1 = g_{ij}\left(\underset{1}{v^k}\dot{\nabla}_k\underset{3}{v^i}\right)\underset{2}{v^j} = -\underset{31}{\overset{2}{\tau}}$. $\square$

Further, by means of Kofoğlu (2020), the following theorem is valid:

Theorem 2. Let C be a curve which has non-zero Bishop curvatures in W_3. C is a slant helix if and only if $\dfrac{k_1}{k_2}$ is constant.

Using Şemin (1983), we can write the following proposition:

Proposition 2. If C is a spherical curve,

$$
\frac{\kappa_2}{\kappa_1} + \underset{1}{v^l}\dot{\nabla}_l\left[\frac{1}{\kappa_2}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{1}{\kappa_1}\right)\right] = 0 \quad (l = 1,2,3)
$$

is satisfied.

Proof. Let C be a spherical curve. If we choose center of the sphere as origin, the position vector at any point of C satisfies the following relation:

$$
\left[g_{ij}x^ix^j\right]^2 = R^2.
\tag{13}
$$

Here R is the radius of the sphere and it is constant.

Taking prolonged covariant derivative of $g_{ij}x^ix^j = R$ in the direction of $\underset{1}{v}$, we get

$$
g_{ij}\underset{1}{v^i}x^j = 0
\tag{14}
$$

where $\underset{1}{v}$ is the tangent vector field of C.

Taking prolonged covariant derivative of (14) in the direction of $\underset{1}{v}$, we have

$$
\begin{aligned}
g_{ij}\left(\underset{1}{v^k}\dot{\nabla}_k\underset{1}{v^i}\right)x^j + 1 &= 0 \\
g_{ij}\kappa_1\underset{2}{v^i}x^j + 1 &= 0 \\
g_{ij}\underset{2}{v^i}x^j &= -\frac{1}{\kappa_1}
\end{aligned}
\tag{15}
$$

where $g_{ij}\underset{1}{v^i}\underset{1}{v^j} = 1$, $\underset{2}{v}$ is the principal normal vector field of C and κ_1 is the first curvature of C.

Taking prolonged covariant derivative of (15) in the direction of $\underset{1}{v}$, we obtain

$$
\begin{aligned}
-\kappa_2 g_{ij}\underset{3}{v^i}x^j &= \underset{1}{v^k}\dot{\nabla}_k\frac{1}{\kappa_1} \\
-g_{ij}\underset{3}{v^i}x^j &= \frac{1}{\kappa_2}\underset{1}{v^k}\dot{\nabla}_k\frac{1}{\kappa_1}
\end{aligned}
\tag{16}
$$

where $\underset{3}{v}$ is the binormal vector field of C, κ_2 is the second curvature of C, $g_{ij}\underset{2}{v^i}\underset{1}{v^j} = 0$ and $g_{ij}\underset{1}{v^i}x^j = 0$.

Again taking prolonged covariant derivative of (16) in the direction of $\underset{1}{v}$, we have

$$
\kappa_2 g_{ij}\underset{2}{v^i}x^j = \underset{1}{v^l}\dot{\nabla}_l\left(\frac{1}{\kappa_2}\underset{1}{v^k}\dot{\nabla}_k\frac{1}{\kappa_1}\right)
\tag{17}
$$

where $g_{ij}\underset{3}{v^i}\underset{1}{v^j} = 0$.

Using (15) in (17), we get

$$
\frac{\kappa_2}{\kappa_1} + \underset{1}{v^l}\dot{\nabla}_l\left(\frac{1}{\kappa_2}\underset{1}{v^k}\dot{\nabla}_k\frac{1}{\kappa_1}\right) = 0
\tag{18}
$$

which concludes the proof. $\square$

With the help of Nomizu and Yano (1974) and Özdeğer and Şentürk (2002), the following definition and proposition can be formulated:

Definition 2. C is called a circle if there exists a vector field z^i and a positive constant k such that

$$\underset{1}{v^k}\dot{\nabla}_k \underset{1}{v^i} = kz^i$$

$$\underset{1}{v^k}\dot{\nabla}_k z^i = -k\underset{1}{v^i}$$

where $g_{ij}z^iz^j = 1$.

Proposition 3. If C is a circle, the equation

$$\underset{1}{v^l}\dot{\nabla}_l\left(\underset{1}{v^k}\dot{\nabla}_k \underset{1}{v^i}\right) + g_{ij}\left(\underset{1}{v^k}\dot{\nabla}_k \underset{1}{v^i}\right)\left(\underset{1}{v^k}\dot{\nabla}_k \underset{1}{v^j}\right)\underset{1}{v^i} = 0$$

is satisfied. Conversely, if C satisfies the above equation, C is either a geodesic or a circle.

4. About The Spherical Images of a Curve in Weyl Space

Definition 3. Let C be a curve in W_3. If we translate the first vector field of type-1 Bishop frame to the center O of the unit sphere S^2, we obtain a spherical image $\bar{C} : u^i = u^i(\bar{s})$ ($\bar{s}$ is the arc length parameter of $\bar{C}$). $\bar{C}$ is called tangent Bishop spherical image or indicatrix of the curve $\bar{C}$.

In order to investigate the relations between type-1 Bishop and Frenet-Serret invariants we take the prolonged covariant derivative of u^i in the direction of $\underset{1}{v}$, we have

$$\underset{1}{v^k}\dot{\nabla}_k u^i = k_1\underset{1}{n^i} + k_2\underset{2}{n^i} \tag{19}$$

$$\left(\underset{1}{\bar{v}^k}\dot{\bar{\nabla}}_k u^i\right) a = k_1\underset{1}{n^i} + k_2\underset{2}{n^i} \tag{20}$$

$$\underset{1}{\bar{v}^i} a = k_1\underset{1}{n^i} + k_2\underset{2}{n^i} \tag{21}$$

where $\underset{1}{\bar{v}^i}$ is the tangent vector field of $\bar{C}$, $g_{ij}\underset{1}{\bar{v}^i}\underset{1}{\bar{v}^j} = 1$ and $a = a(s)$.

Taking the norm of both sides of (21), we get

$$a = \mp\sqrt{k_1^2 + k_2^2}. \tag{22}$$

Let us take $a = \sqrt{k_1^2 + k_2^2}$. Hence, we obtain

$$\underset{1}{\bar{v}^i} = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}\underset{1}{n^i} + \frac{k_2}{\sqrt{k_1^2 + k_2^2}}\underset{2}{n^i} \tag{23}$$

where $k_1 = g_{ij}\underset{1}{c^i}\underset{1}{n^j}$ and $k_2 = g_{ij}\underset{1}{c^i}\underset{2}{n^j}$.

Taking the prolonged covariant derivative of (23) in the direction of $\underset{1}{v}$, we have

$$\begin{aligned}\underset{1}{v^k}\dot{\nabla}_k\underset{1}{\bar{v}^i} &= \left(\underset{1}{\bar{v}^k}\dot{\bar{\nabla}}_k\underset{1}{\bar{v}^i}\right)a \\ &= \frac{k_2^3}{(k_1^2 + k_2^2)^{3/2}}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_1}{k_2}\right)\underset{1}{n^i} \\ &\quad + \frac{k_1^3}{(k_1^2 + k_2^2)^{3/2}}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_2}{k_1}\right)\underset{2}{n^i} \\ &\quad - \sqrt{k_1^2 + k_2^2}\underset{1}{v^i}\end{aligned} \tag{24}$$

and

$$\begin{aligned}\bar{\kappa}_1\underset{2}{\bar{v}^i} &= \frac{k_2^3}{(k_1^2 + k_2^2)^2}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_1}{k_2}\right)\underset{1}{n^i} \\ &\quad + \frac{k_1^3}{(k_1^2 + k_2^2)^2}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_2}{k_1}\right)\underset{2}{n^i} - \underset{1}{v^i}\end{aligned} \tag{25}$$

where $k_1 = g_{ij}\underset{1}{c^i}\underset{1}{n^j}$ and $k_2 = g_{ij}\underset{1}{c^i}\underset{2}{n^j}$.

Taking the norm of both sides of (25), we get

$$\bar{\kappa}_1 = \left(\left[\frac{k_2^3}{(k_1^2 + k_2^2)^2}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_1}{k_2}\right)\right]^2 + \left[\frac{k_1^3}{(k_1^2 + k_2^2)^2}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_2}{k_1}\right)\right]^2 + 1\right)^{1/2} \tag{26}$$

and from (24) and (26)

$$\begin{aligned}\underset{2}{\bar{v}^i} &= \frac{1}{\bar{\kappa}_1}\frac{k_2^3}{(k_1^2 + k_2^2)^2}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_1}{k_2}\right)\underset{1}{n^i} \\ &\quad + \frac{1}{\bar{\kappa}_1}\frac{k_1^3}{(k_1^2 + k_2^2)^2}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_2}{k_1}\right)\underset{2}{n^i} - \frac{1}{\bar{\kappa}_1}\underset{1}{v^i}\end{aligned} \tag{27}$$

where $\bar{\kappa}_1$ is the first curvature of $\bar{C}$, $\underset{2}{\bar{v}^i}$ is the principal vector field of $\bar{C}$, $g_{ij}\underset{2}{\bar{v}^i}\underset{2}{\bar{v}^j} = 1$, $k_1 = g_{ij}\underset{1}{c^i}\underset{1}{n^j}$ and $k_2 = g_{ij}\underset{1}{c^i}\underset{2}{n^j}$.

We know that

$$\underset{3}{\bar{v}^i} = \epsilon_{ijk}\underset{1}{\bar{v}^j}\underset{2}{\bar{v}^k}. \tag{28}$$

Using (23) and (27) in (28), we obtain

$$\begin{aligned}\underset{3}{\bar{v}^i} &= \frac{1}{\bar{\kappa}_1}\Bigg\{\left[\frac{k_1^4}{(k_1^2 + k_2^2)^{5/2}}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_2}{k_1}\right) \right. \\ &\quad \left. - \frac{k_2^4}{(k_1^2 + k_2^2)^{5/2}}\left(\underset{1}{v^k}\dot{\nabla}_k\frac{k_1}{k_2}\right)\right]\underset{1}{v^i} \\ &\quad - \frac{k_2}{\sqrt{k_1^2 + k_2^2}}\underset{1}{n^i} + \frac{k_1}{\sqrt{k_1^2 + k_2^2}}\underset{2}{n^i}\Bigg\}\end{aligned} \tag{29}$$

where $k_1 = g_{ij}\underset{1}{c^i}\underset{1}{n^j}$ and $k_2 = g_{ij}\underset{1}{c^i}\underset{2}{n^j}$.

Taking the prolonged covariant derivative of (29) in the direction of $\underset{1}{v}$ and multiplying this expression by $g_{ij}\underset{2}{\bar{v}^j}$, we get

$$\begin{aligned}\bar{\kappa}_2 &= \frac{1}{(k_1^2 + k_2^2)^3 + \left[k_1^2\underset{1}{v^k}\dot{\nabla}_k\frac{k_2}{k_1}\right]^2} \\ &\quad \Bigg\{\left[\underset{1}{v^l}\dot{\nabla}_l\left(\underset{1}{v^k}\dot{\nabla}_k k_2\right)\right]k_1(k_1^2 + k_2^2) \\ &\quad - \left[\underset{1}{v^l}\dot{\nabla}_l\left(\underset{1}{v^k}\dot{\nabla}_k k_1\right)\right]k_2(k_1^2 + k_2^2) \\ &\quad - 3k_1^2\left(\underset{1}{v^k}\dot{\nabla}_k k_1\right)\left(\underset{1}{v^k}\dot{\nabla}_k k_2\right) \\ &\quad + 3k_1k_2\left(\underset{1}{v^k}\dot{\nabla}_k k_1\right)\left(\underset{1}{v^k}\dot{\nabla}_k k_1\right) \\ &\quad - 3k_1k_2\left(\underset{1}{v^k}\dot{\nabla}_k k_2\right)\left(\underset{1}{v^k}\dot{\nabla}_k k_2\right)\end{aligned} \tag{30}$$

$$+3k_2^2\left(v^k_1\dot{\nabla}_k k_1\right)\left(v^k_1\dot{\nabla}_k k_2\right)\}$$

where $\bar{\kappa}_2$ is the second curvature of $\bar{C}$ and $g_{ij}\bar{v}^i_2\bar{v}^j_2 = 1$, $k_1 = g_{ij}c^i_1 n^j_1$ and $k_2 = g_{ij}c^i_1 n^j_2$.

Corollary 1. Let $\bar{C}$ be the tangent Bishop spherical image of C. If $\frac{k_1}{k_2} = constant$, then the tangent Bishop spherical image $u^i = u^i(\bar{s})$ is a circle.

Proof. If we use Proposition 3, we have

$$v^l_1\dot{\nabla}_l\left(v^k_1\dot{\nabla}_k\bar{v}^i_1\right)$$

$$= \left[v^l_1\dot{\nabla}_l\frac{k_2^3}{\left(k_1^2+k_2^2\right)^{3/2}}\right]\left(v^k_1\dot{\nabla}_k\frac{k_1}{k_2}\right)n^i_1$$

$$+\frac{k_2^3}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}\left[v^l_1\dot{\nabla}_l\left(v^k_1\dot{\nabla}_k\frac{k_1}{k_2}\right)\right]n^i_1$$

$$-\left[v^l_1\dot{\nabla}_l\frac{k_1 k_2^2}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}\right]\left(v^k_1\dot{\nabla}_k\frac{k_1}{k_2}\right)n^i_2$$

$$-\frac{k_1 k_2^2}{\left(k_1^2+k_2^2\right)^{3/2}}v^l_1\dot{\nabla}_l\left(v^k_1\dot{\nabla}_k\frac{k_1}{k_2}\right)n^i_2$$

$$-\frac{1}{\sqrt{k_1^2+k_2^2}}$$

$$\left\{k_2\left(v^l_1\dot{\nabla}_l k_2\right)\frac{(k_1^2+k_2^2)}{k_2^2}+k_2 k_1\left(v^l_1\dot{\nabla}_l\frac{k_1}{k_2}\right)\right\}v^i_1$$

$$-\sqrt{k_1^2+k_2^2}(k_1 n^i_1 + k_2 n^i_2) \qquad (31)$$

and

$$g_{ij}\left(v^k_1\dot{\nabla}_k\bar{v}^i_1\right)\left(v^k_1\dot{\nabla}_k\bar{v}^j_1\right)\bar{v}^i_1$$

$$= \left\{k_2^4\frac{1}{\left(k_1^2+k_2^2\right)^2}\left[v^k_1\dot{\nabla}_k\frac{k_1}{k_2}\right]^2 + \left(k_1^2+k_2^2\right)\right\}\bar{v}^i_1 \qquad (32)$$

where $k_1 = g_{ij}c^i_1 n^j_1$ and $k_2 = g_{ij}c^i_1 n^j_2$.

Under the condition $\frac{k_1}{k_2} = constant$, using (31) and (32), we obtain

$$v^l_1\dot{\nabla}_l\left(v^k_1\dot{\nabla}_k\bar{v}^i_1\right) + g_{ij}\left(v^k_1\dot{\nabla}_k\bar{v}^i_1\right)\left(v^k_1\dot{\nabla}_k\bar{v}^j_1\right)\bar{v}^i_1 = 0. \qquad (33)$$

Hence, we see that, the tangent Bishop spherical image $u^i = u^i(\bar{s})$ of C is a circle. $\square$

Definition 4. Let C be a curve in W_3. If we translate the second vector field of type-1 Bishop frame to the center O of the unit sphere S^2, we obtain a spherical image $\bar{F}: f^i = f^i(\bar{s})$ ($\bar{s}$ is the arc length parameter of $\bar{F}$). $\bar{F}$ is called n_1 Bishop spherical image or indicatrix of the curve C.

The relations between type-1 Bishop and Frenet-Serret invariants are obtained by taking the prolonged covariant derivative of f^i in the direction of v_1, we have

$$v^k_1\dot{\nabla}_k f^i = -k_1 v^i_1 \qquad (34)$$

$$\left(\bar{v}^k_1\dot{\bar{\nabla}}_k f^i\right)b = -k_1 v^i_1 \qquad (35)$$

$$\bar{v}^i_1 b = -k_1 v^i_1 \qquad (36)$$

where $\bar{v}^i_1$ is the tangent vector field of $\bar{F}$, $g_{ij}\bar{v}^i_1\bar{v}^j_1 = 1$ and $b = b(s)$.

Taking the norm of both sides of (36), we get

$$b = \mp k_1. \qquad (37)$$

Let us take $b = -k_1$. In this case, we obtain

$$\bar{v}^i_1 = v^i_1. \qquad (38)$$

Taking the prolonged covariant derivative of (38) in the direction of v_1, we have

$$v^k_1\dot{\nabla}_k\bar{v}^i_1 = \left(\bar{v}^k_1\dot{\bar{\nabla}}_k\bar{v}^i_1\right)b = v^k_1\dot{\nabla}_k v^i_1 \qquad (39)$$

$$\bar{M}_1\bar{v}^i_2(-k_1) = k_1 n^i_1 + k_2 n^i_2 \qquad (40)$$

$$\bar{M}_1\bar{v}^i_2 = -n^i_1 - \frac{k_2}{k_1}n^i_2. \qquad (41)$$

Taking the norm of both sides of (41), we get

$$\bar{M}_1 = \sqrt{1+\left(\frac{k_2}{k_1}\right)^2} = \sqrt{1+\left(\frac{g_{ij}c^i_1 n^j_2}{g_{ij}c^i_1 n^j_1}\right)^2} \qquad (42)$$

and

$$\bar{v}^i_2 = -\frac{1}{\bar{M}_1}n^i_1 - \frac{1}{\bar{M}_1}\frac{k_2}{k_1}n^i_2 \qquad (43)$$

where $\bar{v}^i_2$ is the principal normal vector field of $\bar{F}$, $g_{ij}\bar{v}^i_2\bar{v}^j_2 = 1$ and $\bar{M}_1$ is the first curvature of $\bar{F}$.

$\bar{v}^i_3$ is the binormal vector field of $\bar{F}$ and it is defined in the form:

$$\bar{v}^i_3 = \epsilon_{ijk}\bar{v}^j_1\bar{v}^k_2. \qquad (44)$$

If (38) and (43) are used in (44), we have

$$\bar{v}^i_3 = -\frac{1}{\bar{M}_1}n^i_2 + \frac{1}{\bar{M}_1}\frac{k_2}{k_1}n^i_1 \qquad (45)$$

Taking the prolonged covariant derivative of (45) in the direction of v_1, we get

$$v^k_1 \dot{\nabla}_k \bar{v}^i_3 = \left(\bar{v}^k_1 \dot{\bar{\nabla}}_k \bar{v}^i_3 \right) b = -\bar{M}_2 \bar{v}^i_2 (-k_1)$$
$$= -\left(v^k_1 \dot{\nabla}_k \frac{1}{\bar{M}_1} \right) n^i_2$$
$$+ \left(v^k_1 \dot{\nabla}_k \frac{1}{\bar{M}_1} \right) \frac{k_2}{k_1} n^i_1$$
$$+ \frac{1}{\bar{M}_1} \left(v^k_1 \dot{\nabla}_k \frac{k_2}{k_1} \right) n^i_1 \tag{46}$$

and multiplying (46) by $g_{ij} \bar{v}^j_2$, we obtain

$$\bar{M}_2 = -\frac{1}{[\bar{M}_1]^2} \frac{1}{k_1} \left(v^k_1 \dot{\nabla}_k \frac{k_2}{k_1} \right) = -\frac{k_1}{k_1^2 + k_2^2} v^k_1 \dot{\nabla}_k \frac{k_2}{k_1} \tag{47}$$

or

$$\bar{M}_2 = -\frac{g_{ij} c^i_1 n^j_1}{\left(g_{ij} c^i_1 n^j_1 \right)^2 + \left(g_{ij} c^i_1 n^j_2 \right)^2} v^k_1 \dot{\nabla}_k \left[\frac{g_{ij} c^i_1 n^j_2}{g_{ij} c^i_1 n^j_1} \right] \tag{48}$$

where $\bar{M}_2$ is the second curvature of $\bar{F}$.

Corollary 2. Let $\bar{F}$ be n_1 Bishop spherical image of C. If $\frac{k_1}{k_2} = constant$, then n_1 Bishop spherical image $f^i = f^i(\bar{s})$ is a circle.

Proof. By using Proposition 3, we get

$$v^l_1 \dot{\nabla}_l \left(v^k_1 \dot{\nabla}_k \bar{v}^i_1 \right) = v^l_1 \dot{\nabla}_l \left(v^k_1 \dot{\nabla}_k v^i_1 \right)$$
$$= \left(v^l_1 \dot{\nabla}_l k_1 \right) n^i_1 + \left(v^l_1 \dot{\nabla}_l k_2 \right) n^i_2 \tag{49}$$
$$- (k_1^2 + k_2^2) v^i_1$$

and

$$g_{ij} \left(v^k_1 \dot{\nabla}_k \bar{v}^i_1 \right) \left(v^k_1 \dot{\nabla}_k \bar{v}^j_1 \right) \bar{v}^i_1 = g_{ij} \left(v^k_1 \dot{\nabla}_k v^i_1 \right) \left(v^k_1 \dot{\nabla}_k v^j_1 \right) v^i_1$$
$$= (k_1^2 + k_2^2) v^i_1 \tag{50}$$

Summing (49) and (50), we have

$$v^l_1 \dot{\nabla}_l \left(v^k_1 \dot{\nabla}_k \bar{v}^i_1 \right) + g_{ij} \left(v^k_1 \dot{\nabla}_k \bar{v}^i_1 \right) \left(v^k_1 \dot{\nabla}_k \bar{v}^j_1 \right) \bar{v}^i_1$$
$$= \left(v^l_1 \dot{\nabla}_l k_1 \right) n^i_1 + \left(v^l_1 \dot{\nabla}_l k_2 \right) n^i_2 \tag{51}$$

Using $k_1 = \kappa_1 \cos \theta$ and $k_2 = \kappa_1 \sin \theta$, we obtain

$$v^l_1 \dot{\nabla}_l k_1 = v^l_1 \dot{\nabla}_l (\kappa_1 \cos \theta)$$
$$= \left(v^l_1 \dot{\nabla}_l \kappa_1 \right) \cos \theta - \kappa_1 \left(v^l_1 \dot{\nabla}_l \theta \right) \sin \theta \tag{52}$$

and

$$v^l_1 \dot{\nabla}_l k_2 = v^l_1 \dot{\nabla}_l (\kappa_1 \sin \theta)$$
$$= \left(v^l_1 \dot{\nabla}_l \kappa_1 \right) \sin \theta + \kappa_1 \left(v^l_1 \dot{\nabla}_l \theta \right) \cos \theta \tag{53}$$

where $\theta = \theta(s) = arccot \frac{k_1}{k_2}$ and $v^k_1 \dot{\nabla}_k \theta = \frac{v^k_1 \dot{\nabla}_k \frac{k_1}{k_2}}{1 + \left(\frac{k_1}{k_2} \right)^2}$.

It is known that $\kappa_1 = \sqrt{k_1^2 + k_2^2}$. In this case, we obtain

$$v^k_1 \dot{\nabla}_k \kappa_1 = v^k_1 \dot{\nabla}_k \sqrt{k_2^2 \left[\left(\frac{k_1}{k_2} \right)^2 + 1 \right]}$$
$$= \frac{1}{\sqrt{k_1^2 + k_2^2 - k_2 sin\theta \left(\frac{k_1^2 + k_2^2}{k_2^2} \right)}} \tag{54}$$
$$\cdot \left\{ k_2 \kappa_1 \left(v^k_1 \dot{\nabla}_k \theta \right) cos\theta \left(\frac{k_1^2 + k_2^2}{k_2^2} \right) \right.$$
$$\left. + k_2 k_1 \left(v^k_1 \dot{\nabla}_k \frac{k_1}{k_2} \right) \right\}$$

If $\frac{k_1}{k_2} = constant$, we have $v^k_1 \dot{\nabla}_k \theta = 0$ and $v^k_1 \dot{\nabla}_k \kappa_1 = 0$. This means $v^k_1 \dot{\nabla}_k k_1 = 0$ and $v^k_1 \dot{\nabla}_k k_2 = 0$. Using these results in (51),

$$v^l_1 \dot{\nabla}_l \left(v^k_1 \dot{\nabla}_k \bar{v}^i_1 \right) + g_{ij} \left(v^k_1 \dot{\nabla}_k \bar{v}^i_1 \right) \left(v^k_1 \dot{\nabla}_k \bar{v}^j_1 \right) \bar{v}^i_1 = 0 \tag{55}$$

is obtained. So, it is shown that $\bar{F} : f^i = f^i(\bar{s})$ is a circle. □

Theorem 3. Let $\bar{F} : f^i = f^i(\bar{s})$ be n_1 Bishop spherical image of C. If $\bar{F}$ is a general helix, then

$$\frac{k_1^2}{(k_1^2 + k_2^2)^{3/2}} \left(v^k_1 \dot{\nabla}_k \frac{k_2}{k_1} \right) = constant$$

is valid.

Theorem 4. Let $\bar{F} : f^i = f^i(\bar{s})$ be n_1 Bishop spherical image of C. If $\bar{F}$ is a slant helix, then

$$\frac{(k_1^2 + k_2^2)^4 k_1}{\left\{ (k_1^2 + k_2^2)^3 + k_1^4 \left[v^k_1 \dot{\nabla}_k \frac{k_2}{k_1} \right]^2 \right\}^{3/2}} v^l_1 \dot{\nabla}_l \left[\frac{k_1^2 v^k_1 \dot{\nabla}_k \frac{k_2}{k_1}}{(k_1^2 + k_2^2)^{3/2}} \right] = constant.$$

Since $\bar{F} : f^i = f^i(\bar{s})$ is a spherical curve, by using Proposition 2, we can state the following theorem:

Theorem 5. Let $\bar{F} : f^i = f^i(\bar{s})$ be n_1 Bishop spherical image of C. In this case, the following equation

$$\frac{k_1^2}{(k_1^2 + k_2^2)^{3/2}} \left(v^k_1 \dot{\nabla}_k \frac{k_2}{k_1} \right) - v^k_1 \dot{\nabla}_k \frac{k_1 k_2}{\sqrt{k_1^2 + k_2^2}} = constant.$$

is valid.

Theorem 6. Let $\bar{F} : f^i = f^i(\bar{s})$ be n_1 Bishop spherical image of C. If $\frac{k_2}{k_1} = constant$, then $\bar{v}_3$ is parallel translated along $\bar{v}_1$.

Proof. From (46), it is known that

$$
\underset{1}{v^k} \dot{\nabla}_k \underset{3}{\overline{v}^i} = \left(\underset{1}{\overline{v}^k} \dot{\overline{\nabla}}_k \underset{3}{\overline{v}^i} \right)(-k_1)
$$
$$
= -\left(\underset{1}{v^k} \dot{\nabla}_k \frac{1}{\overline{M}_1} \right) \underset{2}{n^i}
$$
$$
+ \left(\underset{1}{v^k} \dot{\nabla}_k \frac{1}{\overline{M}_1} \right) \frac{k_2}{k_1} \underset{1}{n^i} \qquad (56)
$$
$$
+ \frac{1}{\overline{M}_1} \left(\underset{1}{v^k} \dot{\nabla}_k \frac{k_2}{k_1} \right) \underset{1}{n^i}
$$

and

$$
\underset{1}{\overline{v}^k} \dot{\overline{\nabla}}_k \underset{3}{\overline{v}^i} = \underset{31}{\overline{a}^i} = \frac{1}{k_1} \left(\underset{1}{v^k} \dot{\nabla}_k \frac{1}{\overline{M}_1} \right) \underset{2}{n^i}
$$
$$
- \frac{k_2}{k_1^2} \left(\underset{1}{v^k} \dot{\nabla}_k \frac{1}{\overline{M}_1} \right) \underset{1}{n^i} \qquad (57)
$$
$$
- \frac{1}{k_1} \frac{1}{\overline{M}_1} \left(\underset{1}{v^k} \dot{\nabla}_k \frac{k_2}{k_1} \right) \underset{1}{n^i}
$$

where $\underset{31}{\overline{a}^i}$ is defined $\underset{31}{\overline{a}^i} = \underset{31p}{\overset{p}{\overline{\tau}}} \underset{p}{\overline{v}^i} = \underset{3}{\overline{T}_k} \underset{1}{\overline{v}^k} \underset{p}{\overline{v}^i}$ and it is named the Chebyshev vector field of the first kind of the net $(\underset{1}{\overline{v}}, \underset{2}{\overline{v}}, \underset{3}{\overline{v}})$. Besides, $\underset{31}{\overset{p}{\overline{\tau}}} = \underset{3}{\overline{T}_k} \underset{1}{\overline{v}^k}$ is called the Chebyshev curvature of the first kind of the net $(\underset{1}{\overline{v}}, \underset{2}{\overline{v}}, \underset{3}{\overline{v}})$ (Tsareva and Zlatanov, 1990).

By taking prolong covariant derivative of $\frac{1}{\overline{M}_1}$ in the direction of $\underset{1}{v}$, we get

$$
\underset{1}{v^k} \dot{\nabla}_k \frac{1}{\overline{M}_1} = - \frac{k_1^2 k_2}{(k_1^2 + k_2^2)^{3/2}} \left(\underset{1}{v^k} \dot{\nabla}_k \frac{k_2}{k_1} \right). \qquad (58)
$$

By using (58) in (57) and by considering the condition $\frac{k_2}{k_1} = constant$, we have

$$
\underset{31}{\overline{a}^i} = 0. \qquad (59)
$$

It means that $\underset{3}{\overline{v}}$ is parallel translated along $\underset{1}{\overline{v}}$. $\square$

Definition 5. Let C be a curve in W_3. If we translate the third vector field of type-1 Bishop frame to the center O of the unit sphere S^2, we obtain $\overline{G} : g^i = g^i(\overline{s})$ ($\overline{s}$ is the arc length parameter of $\overline{G}$). $\overline{G}$ is called $\underset{2}{n^i}$ Bishop spherical image or indicatrix of the curve C.

Let us express the relations between type-1 Bishop and Frenet-Serret invariants similarly by taking prolonged covariant derivative of g^i in the direction of $\underset{1}{v}$, we get

$$
\underset{1}{v^k} \dot{\nabla}_k g^i = -k_2 \underset{1}{v^i} \qquad (60)
$$
$$
\left(\underset{1}{\overline{v}^k} \dot{\overline{\nabla}}_k g^i \right) c = -k_2 \underset{1}{v^i} \qquad (61)
$$
$$
\underset{1}{\overline{v}^i} c = -k_2 \underset{1}{v^i}. \qquad (62)
$$

where $\underset{1}{\overline{v}^i}$ is the tangent vector field of $\overline{G}$, $g_{ij} \underset{1}{\overline{v}^i} \underset{1}{\overline{v}^j} = 1$ and $c = c(s)$.

Taking the norm of both sides of (62), we have

$$
c = \mp k_2. \qquad (63)
$$

Let us choose $c = -k_2$. Then, we obtain

$$
\underset{1}{\overline{v}^i} = \underset{1}{v^i}. \qquad (64)
$$

By taking prolonged covariant derivative of (64) in the direction of $\underset{1}{v}$, we have

$$
\underset{1}{v^k} \dot{\nabla}_k \underset{1}{\overline{v}^i} = \left(\underset{1}{\overline{v}^k} \dot{\nabla}_k \underset{1}{\overline{v}^i} \right) c = \underset{1}{v^k} \dot{\nabla}_k \underset{1}{v^i} \qquad (65)
$$
$$
\underset{2}{\overline{H}_1 \overline{v}^i}(-k_2) = k_1 \underset{1}{n^i} + k_2 \underset{2}{n^i} \qquad (66)
$$
$$
\underset{2}{\overline{H}_1 \overline{v}^i} = -\frac{k_1}{k_2} \underset{1}{n^i} - \underset{2}{n^i}. \qquad (67)
$$

By taking the norm of both sides of (67), we get

$$
\overline{H}_1 = \sqrt{\left(\frac{k_1}{k_2} \right)^2 + 1} = \sqrt{\left[\frac{g_{ij} \underset{1}{c^i} \underset{2}{n^j}}{g_{ij} \underset{1}{c^i} \underset{1}{n^j}} \right]^2 + 1} \qquad (68)
$$

and

$$
\underset{2}{\overline{v}^i} = -\frac{1}{\overline{H}_1} \frac{k_1}{k_2} \underset{1}{n^i} - \frac{1}{\overline{H}_1} \underset{2}{n^i}. \qquad (69)
$$

where $\underset{2}{\overline{v}^i}$ is the principal normal vector field of $\overline{G}$, $g_{ij} \underset{2}{\overline{v}^i} \underset{2}{\overline{v}^j} = 1$ and $\overline{H}_1$ is the first curvature of $\overline{G}$.

$\underset{3}{\overline{v}^i}$ is the binormal vector field of $\overline{G}$ and it is expressed as follows:

$$
\underset{3}{\overline{v}^i} = \epsilon_{ijk} \underset{1}{\overline{v}^j} \underset{2}{\overline{v}^k}. \qquad (70)
$$

By using (64) and (69) in (70), we have

$$
\underset{3}{\overline{v}^i} = -\frac{1}{\overline{H}_1} \frac{k_1}{k_2} \underset{2}{n^i} - \frac{1}{\overline{H}_1} \underset{1}{n^i}. \qquad (71)
$$

Taking the prolonged covariant derivative of (71) in the direction of $\underset{1}{v}$, we obtain

$$
\underset{1}{v^k} \dot{\nabla}_k \underset{3}{\overline{v}^i} = \left(\underset{1}{\overline{v}^k} \dot{\overline{\nabla}}_k \underset{3}{\overline{v}^i} \right) c = -\overline{H}_2 \underset{2}{\overline{v}^i}(-k_2)
$$
$$
= -\left(\underset{1}{v^k} \dot{\nabla}_k \frac{1}{\overline{H}_1} \right) \frac{k_1}{k_2} \underset{2}{n^i}
$$
$$
- \frac{1}{\overline{H}_1} \left(\underset{1}{v^k} \dot{\nabla}_k \frac{k_1}{k_2} \right) \underset{2}{n^i} \qquad (72)
$$
$$
+ \left(\underset{1}{v^k} \dot{\nabla}_k \frac{1}{\overline{H}_1} \right) \underset{1}{n^i}
$$

and then multiplying (72) by $g_{ij} \underset{2}{\overline{v}^j}$, we get

$$\overline{H}_2 = \left(\frac{1}{\overline{H}_1}\right)^2 \frac{1}{k_2}\left(\underset{1}{v^k}\dot{\nabla}_k \frac{k_1}{k_2}\right) = \frac{k_2}{k_1^2 + k_2^2}\left(\underset{1}{v^k}\dot{\nabla}_k \frac{k_1}{k_2}\right) \qquad (73)$$

or

$$\overline{H}_2 = \frac{g_{ij}\underset{1}{c^i}\underset{2}{n^j}}{\left(g_{ij}\underset{1}{c^i}\underset{1}{n^j}\right)^2 + \left(g_{ij}\underset{1}{c^i}\underset{2}{n^j}\right)^2}\underset{1}{v^k}\dot{\nabla}_k\left[\frac{g_{ij}\underset{1}{c^i}\underset{1}{n^j}}{g_{ij}\underset{1}{c^i}\underset{2}{n^j}}\right] \qquad (74)$$

where $\overline{H}_2$ is the second curvature of $\overline{G}$.

Corollary 3. Let $\overline{G}$ be $\underset{2}{n}$ Bishop spherical image of C. If $\frac{k_1}{k_2} = constant$, then $\underset{2}{n}$ Bishop spherical image $g^i = g^i(\bar{s})$ is a circle.

Proof. It is the same with the proof of Corollary 2. □

Theorem 7. Let $\overline{G}: g^i = g^i(\bar{s})$ be $\underset{2}{n}$ Bishop spherical image of C. If $\overline{G}$ is a general helix, then

$$\frac{k_2^2}{(k_1^2 + k_2^2)^{3/2}}\left(\underset{1}{v^k}\dot{\nabla}_k \frac{k_1}{k_2}\right) = constant$$

is satisfied.

Theorem 8. Let $\overline{G}: g^i = g^i(\bar{s})$ be $\underset{2}{n}$ Bishop spherical image of C. If $\overline{G}$ is a slant helix, then

$$\frac{k_2(k_1^2 + k_2^2)^4}{\left\{(k_1^2 + k_2^2)^3 + k_1^4\left[\underset{1}{v^k}\dot{\nabla}_k \frac{k_1}{k_2}\right]^2\right\}^{3/2}}\underset{1}{v^l}\dot{\nabla}_l\left[\frac{k_2^2}{(k_1^2 + k_2^2)^{3/2}}\underset{1}{v^k}\dot{\nabla}_k \frac{k_1}{k_2}\right]$$
$$= constant$$

is valid.

Since $\overline{G}: g^i = g^i(\bar{s})$ is spherical curve, with the help of Proposition 2, we can state the following theorem:

Theorem 9. Let $\overline{G}: g^i = g^i(\bar{s})$ be $\underset{2}{n}$ Bishop spherical image of C. In that case, the following equation

$$\frac{k_2^2}{(k_1^2 + k_2^2)^{3/2}}\left(\underset{1}{v^k}\dot{\nabla}_k \frac{k_2}{k_1}\right) - \underset{1}{v^k}\dot{\nabla}_k \frac{k_1 k_2}{\sqrt{k_1^2 + k_2^2}} = constant$$

is satisfied.

Theorem 10. Let $\overline{G}: g^i = g^i(\bar{s})$ be $\underset{2}{n}$ Bishop spherical image of C. If $\frac{k_1}{k_2} = constant$, then $\underset{3}{\bar{v}}$ is parallel translated along $\underset{1}{\bar{v}}$.

Proof. With reason of equation (72), the following equality can be written:

$$\underset{1}{\bar{v}^k}\dot{\overline{\nabla}}_k\underset{3}{\bar{v}^i} = \underset{31}{\bar{a}^i} = -\frac{k_1}{k_2^2}\left(\underset{1}{v^k}\dot{\nabla}_k \frac{1}{\overline{H}_1}\right)\underset{2}{n^i}$$
$$- \frac{1}{k_2}\frac{1}{\overline{H}_1}\left(\underset{1}{v^k}\dot{\nabla}_k \frac{k_1}{k_2}\right)\underset{2}{n^i} \qquad (75)$$
$$+ \frac{1}{k_2}\left(\underset{1}{v^k}\dot{\nabla}_k \frac{1}{\overline{H}_1}\right)\underset{1}{n^i}.$$

Taking the prolonged covariant derivative of $\frac{1}{\overline{H}_1}$ in the direction of $\underset{1}{v}$, we get

$$\underset{1}{v^k}\dot{\nabla}_k \frac{1}{\overline{H}_1} = -\frac{k_1 k_2^2}{(k_1^2 + k_2^2)^{3/2}}\left(\underset{1}{v^k}\dot{\nabla}_k \frac{k_1}{k_2}\right). \qquad (76)$$

Using (76) in (75), under the condition $\frac{k_1}{k_2} = constant$, we obtain

$$\underset{31}{\bar{a}^i} = \mathbf{0} . \qquad (77)$$

So, we see that $\underset{3}{\bar{v}}$ is parallel translated along $\underset{1}{\bar{v}}$. □

5. Conclusions

In this work, we have defined some special curves, such as general helix, slant helix, spherical curve and circle in Weyl space, by using prolonged covariant derivative. Later, we have examined the spherical images of a curve and the conditions to be special curves of these images. Further, we expressed the relations among the vector fields of Frenet-Serret and Bishop frame belonging to the images. We have seen that, all of the concepts discussed can be written in terms of Bishop curvatures. Earlier, the equivalents in Weyl space of Bishop curvatures were obtained. Thus, they were used in this discussion. Finally, parallel displacement condition for binormal vector fields of $\underset{1}{n}$ and $\underset{2}{n}$ spherical images was formulated.

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